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SIR ASUTOSH MOOKERJEE
SILVER JUBILEE
VOLUMES.

VOL. II.
SCIENCE.

Sie hüllt den Menschen in Dumpfheit ein und spornt ihn ewig zum Lichte. Sie macht ihn abhängig zur Erde, träg und schwer und schüttelt ihn immer wieder auf.—Goethe.

[She, (Nature), wraps man in darkness, and makes him for ever long for light. She creates him dependent upon the earth, dull and heavy ; and yet is always shaking him until he attempts to soar above it.]

Science moves, but slowly, slowly, creeping on from point to point.—*Tennyson.*

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TO THE HONOURABLE SIR ASUTOSH MOOKERJEE, KT., C.S.I.,

ON THE OCCASION OF THE SILVER JUBILEE OF HIS ATTAINING THE

DEGREE OF DOCTOR OF LAWS OF THE UNIVERSITY OF CALCUTTA,

THIS VOLUME OF ESSAYS CONTRIBUTED BY HIS FRIENDS AND

ADMIRERS IN INDIA AND ABROAD IS INSCRIBED WITH

AFFECTION, RESPECT AND GRATITUDE FOR HIS

SPLENDID SERVICES TO THE CAUSE OF

THE ADVANCEMENT OF LEARNING.

1894—1919.

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HEMIPTERIC MATRICES.

By C. E. Cullis, M.A., Ph.D.,

Hardinge Professor of Mathematics, University of Calcutta.

[Art. 1. Representations of a compound matrix.
Art. 2. Simple reversants; symbolic commutants; commutantal products of simple matrices.
Art. 3. Compound reversants; commutantal products of compound matrices.
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1. Representations of a compound matrix.

1. *Standard notations for a compound matrix.*

If $\alpha_1, \alpha_2, \dots \alpha_r$ and $\beta_1, \beta_2, \dots \beta_s$ are any two sets of positive integers, and if

$$\alpha_1 + \alpha_2 + \dots + \alpha_r = m, \quad \beta_1 + \beta_2 + \dots + \beta_s = n, \tag{1}$$

a matrix $X=[x]^n_m$ expressed in one of the equivalent forms

$$X=\begin{bmatrix} X_{11}, X_{12}, \dots X_{1s} \\ X_{21}, X_{22}, \dots X_{2s} \\ \dots\dots\dots\dots \\ X_{r1}, X_{r2}, \dots X_{rs} \end{bmatrix}^{\beta_1, \beta_2, \dots \beta_s}_{\alpha_1, \alpha_2, \dots \alpha_r} = \begin{array}{c|ccc|} & \beta_1 & \beta_2 & \beta_s \\ \alpha_1 & X_{11}, & X_{12}, \dots & X_{1s} \\ \alpha_2 & X_{21}, & X_{22}, \dots & X_{2s} \\ & \dots & \dots & \dots \\ \alpha_r & X_{r1}, & X_{r2}, \dots & X_{rs} \end{array}, \tag{2}$$

where the i th horizontal minor contains α_i horizontal rows and the j th vertical minor contains β_j vertical rows, will be called a compound matrix of the *class*

$$M\begin{pmatrix}\beta_1, \beta_2, \dots \beta_s \\ \alpha_1, \alpha_2, \dots \alpha_r\end{pmatrix} \tag{3}$$

in which the horizontal and vertical *index numbers* are $\alpha_1, \alpha_2, \dots \alpha_r$ and $\beta_1, \beta_2, \dots \beta_s$. The first representation of X in (2) is the ordinary standard notation for a compound matrix; the alternative standard notation on the right will be convenient on account of its conciseness when a large number of constituents are shown. Any matrix $X=[x]_m^n$ whose horizontal and vertical orders are m and n can be expressed as a compound matrix of the class (3) where $\alpha_1, \alpha_2, \ldots \alpha_r$ and $\beta_1, \beta_2, \ldots \beta_s$ are any positive integers satisfying the relations (2). When the class of X is taken to be $M\binom{n}{m}$, X is not regarded as compound, and will be called a simple matrix.

2. *Representations of a compound matrix by a skeleton, class-symbol or scheme.*

When a compound matrix $X=[x]_m^n$ of the class (3) is expressed in the forms (2), we will define the *skeleton* of X to be the matrix

$$[X]_r^s = \begin{bmatrix} {}^{1,2,\ldots s} \\ X \\ {}_{1,2,\ldots r} \end{bmatrix} = \left| \begin{matrix} X_{11} & X_{12} & \ldots & X_{1s} \\ X_{21} & X_{22} & \ldots & X_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ X_{r1} & X_{r2} & \ldots & X_{rs} \end{matrix} \right| \tag{4}$$

formed by omitting the index numbers in (2); and we will define

$$\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ \alpha_1, \alpha_2, \ldots \alpha_r \end{pmatrix} \tag{5}$$

to be the *class-symbol* of X, the class-symbol being formed with the successive horizontal and vertical index numbers of X. The compound matrix X is formed from its skeleton by replacing the element X_{ij} of the skeleton by the constituent $[X_{ij}]_{\alpha_i}^{\beta_j}$, which is a matrix with α_i horizontal and β_j vertical rows, this being true for all the values $1, 2, \ldots r$ of i and all the values $1, 2, \ldots s$ of j. The complete representations of X in the forms (2) are known when the skeleton (4) and the class-symbol (5) are known. The skeleton gives the notations used for the constituents of X, and the class-symbol shows what the horizontal and vertical orders of each constituent are. We can represent X by its skeleton only when the class-symbol is known; and we can represent X by its class-symbol only when the notations for its constituents are known or are left undefined.

In our language we shall usually identify the constituents of X with the elements of the skeleton, and speak of the lines of the skeleton as lines of X. Those consecutive constituents X_{ij} in (2) which in the skeleton form lines sloping diagonally from above downwards and towards the right (or towards the left) will be regarded as forming the *diagonal lines* (or the *counter-diagonal lines*) of X. The *diagonals* of X are the diagonal lines drawn through the corner constituents X_{11} and X_{rs}; the *counter-diagonals* of X are the counter-diagonal lines drawn through the corner constituents X_{1s} and X_{r1}. The *leading diagonal* of X is the diagonal through the top left-hand constituent X_{11}. By the *diagonal constituents* of X we usually mean the constituents forming the leading diagonal. When the number of horizontal index numbers is equal to the number of vertical index numbers, i.e. when $r=s$, there is only one diagonal and only one counter-diagonal; and in this case the term 'diagonal constituent' is entirely free from ambiguity.

A *super-minor* of X is a minor formed with complete constituents, and will often be regarded as a minor of the skeleton (3). The super-minor

$$\begin{bmatrix} X_{u\lambda}, & X_{u\mu}, & \dots & X_{u\nu} \\ X_{v\lambda}, & X_{v\mu}, & \dots & X_{v\nu} \\ \dots & \dots & \dots & \dots \\ X_{w\lambda}, & X_{w\mu}, & \dots & X_{w\nu} \end{bmatrix} \begin{matrix} \beta_\lambda, \; \beta_\mu, \; \dots \; \beta_\nu \\ \\ \\ \\ \alpha_u, \; \alpha_v, \; \dots \; \alpha_w \end{matrix}$$

will usually be denoted by the most suitable of the expressions

$$\begin{pmatrix} \beta_\lambda, & \beta_\mu, & \dots & \beta_\nu \\ \alpha_u, & \alpha_v, & \dots & \alpha_w \end{pmatrix}, \quad \begin{bmatrix} \lambda, \mu, \dots \nu \\ X \\ u, v, \dots w \end{bmatrix}, \quad \begin{bmatrix} \lambda, \mu, \dots \nu \\ u, v, \dots w \end{bmatrix},$$

which are respectively its *class-symbol*, its *skeleton*, and its *skeleton-symbol*. An *unbroken* super-minor is a corranged super-minor formed by the intersections of *consecutive* horizontal and *consecutive* vertical minors. Thus the super-minor shown above is unbroken when $u, v, \dots w$ and $\lambda, \mu, \dots \nu$ are two series of consecutive positive integers arranged in ascending orders of magnitude.

When a compound matrix $X = [x]_m^n$ is expressed in the forms (2), we define its *scheme* to be formed by replacing each symbol X_{ij} by 0 or by a cross according as the corresponding constituent of X is or is not known to be a zero matrix. The *skeleton-scheme* of X is formed in the same way from the skeleton (3). We use such schemes to show concisely the general character of a compound matrix. When a constituent X_{ij} is known to be a zero matrix, we shall often replace X_{ij} by 0 both in the expressions (2) and in the skeleton (3).

The notations for the elements of the skeleton of a compound matrix X, i.e. for the constituents of X, may be any we please. Thus the complete representations of a compound matrix X whose class-symbol and skeleton are

$$\begin{pmatrix} \alpha, \beta, \gamma \\ \lambda, \mu \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} {}^{p,q,r} \\ X \\ {}_{u,v} \end{bmatrix} = \begin{bmatrix} X_{up}, & X_{uq}, & X_{ur} \\ X_{vp}, & X_{vq}, & X_{vr} \end{bmatrix}$$

are

$$X = \begin{bmatrix} X_{up}, & X_{uq}, & X_{ur} \\ X_{vp}, & X_{vq}, & X_{vr} \end{bmatrix}_{\lambda, \mu}^{\alpha, \beta, \gamma} = \begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \lambda & X_{up}, & X_{uq}, & X_{ur} \\ \mu & X_{vp}, & X_{vq}, & X_{vr} \end{array}$$

3. *Zero index numbers.*

The index numbers of a compound matrix X are usually considered to be all different from 0 ; but this is not necessary. When any horizontal (or vertical) index number is 0, the corresponding horizontal (or vertical) minor of X is absent, or can be struck out. Consequently we can always add to or remove from the index numbers as many 0's as we please occupying any positions amongst them. When we do this, the matrix X and its class remain unaltered, but we alter the skeleton and the class-symbol, and we therefore alter the *designations* of the actually occurring constituents; for the number of horizontal or vertical rows in the skeleton is always equal to the total number of horizontal or vertical index numbers, 0's included.

When the number of vertical minors is equal to the number of horizontal minors, or has been made equal to it by the addition or removal of zero index numbers, the class-symbol (5) assumes the form

$$\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_r \\ \alpha_1, \alpha_2, \ldots \alpha_r \end{pmatrix}, \tag{5'}$$

and then the diagonal constituents (some or all of which may be absent), are those of orders $\begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix}$, $\begin{pmatrix} \beta_2 \\ \alpha_2 \end{pmatrix}$, ... $\begin{pmatrix} \beta_r \\ \alpha_r \end{pmatrix}$. By varying the zero index numbers and their positions we can vary the constituents which are to be regarded as diagonal constituents. When we use the class-symbol (5) in which $s \neq r$, we usually imagine $s - r$ final 0's to be added to the horizontal index numbers or $r - s$ final 0's to be added to the vertical index numbers according as $s - r$ or $r - s$ is positive. This enables us to regard the constituents of the leading diagonal as the diagonal constituents. The following are different representations of a given compound matrix of the class $M\begin{pmatrix} 2, 4, 3 \\ 3, 2 \end{pmatrix}$:

	Class symbol.	Skeleton.	Representation of X.
(1)	$\begin{pmatrix}2,4,3\\3,2\end{pmatrix}$	$\begin{bmatrix}X\end{bmatrix}^{1,2,3}_{1,2}$	$X=\begin{bmatrix}X_{11}, & X_{12}, & X_{13}\\X_{21}, & X_{22}, & X_{23}\end{bmatrix}^{2,4,3}_{3,2}$
(2)	$\begin{pmatrix}2,4,3\\3,2,0\end{pmatrix}$	$\begin{bmatrix}X\end{bmatrix}^{1,2,3}_{1,2,3}$	$X=\begin{bmatrix}X_{11}, & X_{12}, & X_{13}\\X_{21}, & X_{22}, & X_{23}\end{bmatrix}^{2,4,3}_{3,2}$
(3)	$\begin{pmatrix}2,4,3\\0,3,2\end{pmatrix}$	$\begin{bmatrix}X\end{bmatrix}^{1,2,3}_{1,2,3}$	$X=\begin{bmatrix}X_{21}, & X_{22}, & X_{23}\\X_{31}, & X_{32}, & X_{33}\end{bmatrix}^{2,4,3}_{3,2}$
(4)	$\begin{pmatrix}0,2,4,3,0\\0,0,3,2,0\end{pmatrix}$	$\begin{bmatrix}X\end{bmatrix}^{1,2,3,4,5}_{1,2,3,4,5}$	$X=\begin{bmatrix}X_{32}, & X_{33}, & X_{34}\\X_{42}, & X_{43}, & X_{44}\end{bmatrix}^{2,4,3}_{3,2}$

Here only the actually occurring constituents have been shown; and these are the same in all cases, though their designations are different. In (2) there are 3 diagonal constituents X_{11}, X_{22}, X_{33} of which the last is absent; in (3) there are 3 diagonal constituents X_{11}, X_{22}, X_{33} of which the first is absent; in (4) there are 5 diagonal constituents X_{11}, X_{22}, X_{33}, X_{44}, X_{55} of which the first two and the last are absent. We usually regard (1) and (2) as the same.

2. Simple reversants; symbolic commutants; commutantal products of simple matrices.

1. *Simple reversants.*

For the undegenerate symmetric matrix formed by reversing the order of arrangement of the horizontal or vertical rows of the unit matrix $[1]^r_r$ we shall use the notation

$$[j]^r_r=\begin{vmatrix}0\ldots0 & 1\\0\ldots1 & 0\\\ldots\ldots & \\1\ldots0 & 0\end{vmatrix},\ \text{which includes } [j]^1_1=[1], \tag{A}$$

and we will call $[j]^r_r$ the simple reversant of order r or the reversant of the class $\mathrm{M}\binom{r}{r}$. When this matrix is applied as a prefactor to a matrix A with r horizontal rows, it reverses the order of arrangement of the horizontal rows of A; and when it is applied as a postfactor to a matrix A with r vertical rows, it reverses the order of arrangement of the vertical rows of A. The equations show that $[j]_r^r$ is its own inverse, and that its reciprocal and conjugate reciprocal are the undegenerate symmetric matrix

$$(-1)^{\frac{1}{2}r(r-1)}\,[j]_r^r.$$

2. *Symbolic commutants.*

Taking π and π' to be two different symbolic elements, we will impose on the four expressions

$$\{\pi, \pi\},\quad \{\pi', \pi'\},\quad \{\pi', \pi\},\quad \{\pi, \pi'\} \tag{B}$$

the law of composition of the symbols for commutants, and call them *symbolic commutants*. A product

$$\alpha\beta\gamma\ldots\kappa\lambda \tag{1}$$

in which each of the factors is one of the four symbolic commutants (B) is *commutantal* (in the symbolic sense) when in every pair of consecutive factors the second element of the factor on the left is the same as the first element of the factor on the right. Whenever the product (1) is commutantal, it is to be regarded as equivalent to that one ξ of the four symbolic commutants (B) which is such that the first element of ξ is the first element of the first factor symbol α, and the second element of ξ is the second element of the last factor symbol λ. We then call ξ the *product symbol* of the symbolic commutantal product (1) and put

$$\alpha\beta\gamma\ldots\kappa\lambda = \xi. \tag{2}$$

Consistently with the above definitions we can define j to be a symbolic reversant subject to the relation

$$jj = 1, \tag{B$_1$}$$

which when applied as a prefactor to any one of the symbolic commutants (B) changes its first element from π to π' or from π' to π, and which when applied as a postfactor to any one of them changes its second element from π to π' or from π' to π. Then the four symbolic commutants (B) are subject to the relations

$$\begin{aligned}
\{\pi', \pi'\} &= j\{\pi, \pi\}j, & \{\pi', \pi\} &= j\{\pi, \pi\}, & \{\pi, \pi'\} &= \{\pi, \pi\}j,\\
\{\pi, \pi\} &= j\{\pi', \pi'\}j, & \{\pi, \pi'\} &= j\{\pi', \pi'\}, & \{\pi', \pi\} &= \{\pi', \pi'\}j,\\
\{\pi, \pi'\} &= j\{\pi', \pi\}j, & \{\pi, \pi\} &= j\{\pi', \pi\}, & \{\pi', \pi'\} &= \{\pi', \pi\}j,\\
\{\pi', \pi\} &= j\{\pi, \pi'\}j, & \{\pi', \pi'\} &= j\{\pi, \pi'\}, & \{\pi, \pi\} &= \{\pi, \pi'\}j,
\end{aligned} \tag{B$_2$}$$

$$\{\pi, \pi\}\{\pi, \pi\}\ldots\{\pi, \pi\} = \{\pi, \pi\}, \tag{B$_3$}$$

the remaining 3 lines in (B_2) being deducible from any one of the 4 lines by use of the relation (B_1); and the value of any commutantal product of the symbolic commutants (B) can be deduced from that of the product (B_3) by using the relations (B_1) and (B_2). If $\xi_1, \xi_2, \xi_3, \xi_4$ are the 4 symbolic commutants (B) taken in any order, the 12 relations (B_2) are all included in the 16 relations given by the formula

$$\xi_k = a\xi_h b, \tag{C}$$

where a is 1 or j according as the first elements of ξ_h and ξ_k are like or unlike, and b is 1 or j according as the second elements of ξ_h and ξ_k are like or unlike. When a or b is 1, we will regard it as being absent.

Ex. i. The product $\{\pi, \pi'\}\{\pi', \pi'\}\{\pi', \pi\}\{\pi, \pi'\} = \{\pi, \pi'\}$ is commutantal. If we denote the product symbol by ξ, and evaluate it by using the relations (B_1), (B_2), (B_3), we have

$$\begin{aligned}\xi = \{\pi, \pi\}\, j \,.\, j\, \{\pi, \pi\}\, j \,.\, j\, \{\pi, \pi\} \,.\, \{\pi, \pi\}\, j &= \{\pi, \pi\}\{\pi, \pi\}\{\pi, \pi\}\{\pi, \pi\} \,.\, j \\ &= \{\pi, \pi\}\, j = \{\pi, \pi'\}.\end{aligned}$$

Ex. ii. If ξ is any one of the symbolic commutants (B), then the four symbolic commutants are

$$\xi,\ j\xi j,\ j\xi,\ \xi j.$$

3. *Simple matrices correlated by simple reversants.*

Let $X = [x]_m^n$ be any simple matrix of the class $M\binom{n}{m}$. Then if X', Y, Y' are the matrices derived from X by reversing the orders of arrangement of both its horizontal and vertical rows, the order of arrangement of its horizontal rows only, the order of arrangement of its vertical rows only, we have

$$X = \begin{bmatrix} x_{11} \ldots x_{1n} \\ \ldots\ldots\ldots \\ x_{m1} \ldots x_{mn} \end{bmatrix},\quad X' = \begin{bmatrix} x_{mn} \ldots x_{m1} \\ \ldots\ldots\ldots \\ x_{1n} \ldots x_{11} \end{bmatrix},\quad Y = \begin{bmatrix} x_{m1} \ldots x_{mn} \\ \ldots\ldots\ldots \\ x_{11} \ldots x_{1n} \end{bmatrix},\quad Y' = \begin{bmatrix} x_{1n} \ldots x_{11} \\ \ldots\ldots\ldots \\ x_{mn} \ldots x_{m1} \end{bmatrix}; \tag{B'}$$

and if $J_m = [j]_m^m$ and $J_n = [j]_n^n$ are the simple reversants of orders m and n, so that

$$J_m J_m = [1]_m^m, \quad J_n J_n = [1]_n^n, \tag{B_1'}$$

these four matrices are connected by the relations

$$\begin{aligned} X' &= J_m X J_n, & Y &= J_m X, & Y' &= X J_n, \\ X &= J_m X' J_n, & Y' &= J_m X', & Y &= X' J_n, \\ Y' &= J_m Y J_n, & X &= J_m Y, & X' &= Y J_n, \\ Y &= J_m Y' J_n, & X' &= J_m Y', & X &= Y' J_n, \end{aligned} \tag{B_2'}$$

where each of the remaining 3 lines can be deduced from any one of the 4 lines by using the relations (B_1'). The four matrices (B') will be said to be *correlated* by simple reversants. When any one of them is given the relations (B_2') determine the other three. If X_1, X_2, X_3, X_4 are these four correlated matrices taken in any order, the 12 relations (B_2') are all included in the 16 relations given by the formula

$$X_k = AX_hB, \qquad (C')$$

where A is $[1]_m^m$ or J_m according as the horizontal rows of X_h and X_k have the same or reversed arrangements, and B is $[1]_n^n$ or J_n according as the vertical rows of X_h and X_k have the same or reversed arrangements. When A is $[1]_m^m$, we may regard it as being absent; and when B is $[1]_n^n$, we may regard it as being absent.

We can choose such an arrangement $\xi_1, \xi_2, \xi_3, \xi_4$ of the 4 symbolic commutants (B) that the relations (C′) become the relations (C) when X_1, X_2, X_3, X_4 are replaced respectively by $\xi_1, \xi_2, \xi_3, \xi_4$, and each of the simple reversants J_m and J_n is replaced by the symbolic reversant j; whilst conversely the relations (C) become the relations (C′) when $\xi_1, \xi_2, \xi_3, \xi_4$ are replaced respectively by X_1, X_2, X_3, X_4, the prefactor j is replaced by the simple reversant J_m reversing the order of arrangement of the horizontal rows, and the postfactor j is replaced by the simple reversant J_n reversing the order of arrangement of the vertical rows. We can then distinguish between the four correlated matrices X_1, X_2, X_3, X_4 by ascribing to them the respective commutantal *types* $\xi_1, \xi_2, \xi_3, \xi_4$; regarding the symbolic commutants $\xi_1, \xi_2, \xi_3, \xi_4$ and the symbolic commutantal equations (C) as representatives of the correlated matrices X_1, X_2, X_3, X_4 and the equations (C′). We can ascribe to any one of the four correlated matrices (B′) any one of the four types (B), and the types of the other 3 correlated matrices are then uniquely determinate. The actual values of all four of the correlated matrices are known from their types when the actual value of any one of them is given; for a change in the first (or second) element of ξ_h corresponds to and represents a reversal of the horizontal (or vertical) rows of X_h.

Ex. iii. If ξ, ξ', η, η' are the types ascribed to X, X', Y, Y', the four possible sets of values of ξ, ξ', η, η' are

(1) $\xi = \{\pi, \pi\}$, $\xi' = \{\pi', \pi'\}$, $\eta = \{\pi', \pi\}$, $\eta' = \{\pi, \pi'\}$;

(2) $\xi = \{\pi', \pi'\}$, $\xi' = \{\pi, \pi\}$, $\eta = \{\pi, \pi'\}$, $\eta' = \{\pi', \pi\}$;

(3) $\xi = \{\pi', \pi\}$, $\xi' = \{\pi, \pi'\}$, $\eta = \{\pi, \pi\}$, $\eta' = \{\pi, \pi'\}$;

(4) $\xi = \{\pi, \pi'\}$, $\xi' = \{\pi', \pi\}$, $\eta = \{\pi', \pi'\}$, $\eta' = \{\pi, \pi\}$.

Since π and π' are two arbitrary symbols, these four choices are essentially the same.

Ex. iv. If the types ascribed to X_h and X_k are $\{\pi', \pi'\}$ and $\{\pi, \pi'\}$, the relation $\{\pi, \pi'\} = j\{\pi', \pi'\}$ shows that $X_k = J_m X_h$, *i.e.* X_k is derived from X_h by reversing the order of arrangement of the horizontal rows.

4. *Commutantal products of simple matrices.*

A product of any number of simple matrices will be called a *commutantal product* (in the symbolic sense) when:

(1) it is in standard form, so that in every pair of consecutive factors the number of vertical rows in the factor on the left is equal to the number of horizontal rows in the factor on the right;

(2) such types are ascribed to the factors that the product of the symbolic commutants which represent their types is commutantal.

Thus the product $ABCD = \lfloor a \rfloor_m^p \lfloor b \rfloor_p^q \lfloor c \rfloor_q^r \lfloor d \rfloor_q^n$ is commutantal when the types $\alpha = \{\pi, \pi'\}$, $\beta = \{\pi', \pi'\}$, $\gamma = \{\pi', \pi\}$, $\delta = \{\pi, \pi'\}$ are ascribed to the factors A, B, C, D; because then the product

$$\alpha\beta\gamma\delta = \{\pi, \pi'\}\{\pi', \pi'\}\{\pi', \pi\}\{\pi, \pi'\}$$

is commutantal.

Let
$$\lfloor a \rfloor_m^p \lfloor b \rfloor_p^q \lfloor c \rfloor_q^r \ldots \lfloor k \rfloor_v^w \lfloor l \rfloor_w^n = \lfloor x \rfloor_m^n \quad \text{or} \quad ABC \ldots KL = X \tag{3}$$

be any commutantal product of τ simple matrices in which the product of the symbolic commutants representing the types ascribed to the factors $A, B, C, \ldots K, L$ is

$$\alpha\beta\gamma \ldots \kappa\lambda = \xi, \text{ where } \xi \text{ is a known symbolic commutant.} \tag{4}$$

By replacing each of the factors $A, B, \ldots L$ in (3) by one of the four matrices correlated with it (of which it itself is one) we can form 4^τ different products of which $2^{\tau+1}$ are commutantal, the commutantal products being those in which corresponding reversals are applied to corresponding passive rows. Let

$$\lfloor a' \rfloor_m^p \lfloor b' \rfloor_p^q \lfloor c' \rfloor_q^r \ldots \lfloor k' \rfloor_v^w = \lfloor x' \rfloor_m^n \quad \text{or} \quad A'B'C' \ldots K'L' = X' \tag{3'}$$

be any one of these $2^{\tau+1}$ commutantal products correlated with (3), the product of the symbolic commutants representing the types of $A', B', C', \ldots K', L'$ being

$$\alpha'\beta'\gamma' \ldots \kappa'\lambda' = \xi', \text{ where } \xi \text{ is a known symbolic commutant.} \tag{4'}$$

Because the reversals applied to the passive rows in (3) in deriving (3′)

from it are replaceable in pairs by unit matrices placed between adjacent factors, we have

$$X' = PXQ, \quad \xi' = p\xi_q,$$

where P is the prefactor applied to A, which is either $[1]_m^m$ or $[j]_m^m$, and Q is the postfactor applied to L, which is either $[1]_n^n$ or $[j]_n^n$, and where p is either 1 or j, and q is either 1 or j. Moreover by the definitions of the product symbols ξ and ξ' we have $p = j$ when and only when $P = [j]_m^m$, and $q = j$ when and only when $Q = [j]_n^n$; and we conclude that:

If we ascribe to the product matrix X in (3) the type ξ, then the product matrix X' in (3′) is the matrix of type ξ' correlated with X.

Thus when the product matrix in the commutantal product (3) is known, the product matrices in all the correlated commutantal products derived from (3) by simple reversals are known by simply observing the types of the corresponding products of symbolic commutants, and there is no need to evaluate them directly.

We will call (3) a *commutantal equation* of the type (4) when the product on the left in (3) is commutantal, and the types ascribed to $A, B, C, \dots K, L$ and X are $\alpha, \beta, \gamma, \dots \kappa, \lambda$ and ξ. A commutantal equation of the form

$$HXK = Y$$

will be called a *commutantal transformation* converting X into Y.

Ex. v. Let $[a]_m^p\,[b]_p^q\,[c]_q^r\,[d]_r^s\,[e]_s^n = [x]_m^n$ or $ABCDE = X$ (5)

be a given commutantal product, the product of the symbolic commutants representing the types ascribed to A, B, C, D, E being

$$\{\pi, \pi'\}\{\pi', \pi\}\{\pi, \pi\}\{\pi, \pi'\}\{\pi', \pi'\} = \xi, \quad \text{where} \quad \xi = \{\pi, \pi'\};$$

and let $[a']_m^p\,[b']_p^q\,[c']_q^r\,[d']_r^s\,[e']_s^n = [x']_m^n$ or $A'B'C'D'E' = X'$ (5′)

be a correlated commutantal product obtained by applying simple reversals to the horizontal and vertical rows of some of the factors in (5), the product of the symbolic commutants representing the types of A', B', C', D', E' being

$$\{\pi', \pi'\}\{\pi', \pi\}\{\pi, \pi'\}\{\pi', \pi\}\{\pi, \pi'\} = \xi', \quad \text{where} \quad \xi' = \{\pi', \pi'\}.$$

In order to evaluate the new product matrix X' directly, we should define $J_m, J_p, J_q, J_r, J_s, J_n$ to be the simple reversants of orders m, p, q, r, s, n so that

$$A' = J_mA,\ B' = B,\ C' = CJ_r,\ D' = J_rDJ_s,\ E' = J_sE.$$

Then $X' = J_mA \,.\, B \,.\, CJ_r \,.\, J_rDJ_s \,.\, J_sE = J_m \,.\, ABCDE = J_mX;$ (6)

i.e. X' is the matrix obtained by reversing the order of arrangement of the horizontal rows of X.

Now the equation (6) corresponds to and can be represented by the equation

$$\xi' = j\{\pi, \pi'\} \,.\, \{\pi', \pi\} \,.\, \{\pi, \pi\} j \,.\, j\{\pi, \pi'\} j \,.\, j\{\pi' \; \pi'\} = j\,\xi\,; \tag{6'}$$

and the equation $X' = J_m X$ can be deduced from the equation $\xi' = j\,\xi$. Therefore the equation $\xi' = j\,\xi$ is sufficient to show the relation between X' and X; and if we ascribe to X the type $\xi = \{\pi, \pi'\}$, we know that X' is the matrix of type $\xi' = \{\pi', \pi'\}$ correlated with X.

3. Compound reversants; commutantal products of compound matrices.

1. *Part-reversants.*

Let $\alpha_1, \alpha_2, \ldots \alpha_r$ and $\beta_1, \beta_2, \ldots \beta_s$ be two sets of positive integers such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_r = m, \quad \beta_1 + \beta_2 + \ldots + \beta_s = n\,; \tag{1}$$

let the standard notations of Art. **2** be used for simple reversants, and let $X = [x]_m^n$ be any compound matrix of the class

$$M\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ \alpha_1, \alpha_2, \ldots \alpha_r \end{pmatrix} \tag{2}$$

expressed in the forms (2) of Art. **1**. Then the compound square matrices

$$J_m = \begin{array}{c|c|} & \alpha_1 \;\; \alpha_2 \;\;\;\; \alpha_r \\ \begin{matrix} \alpha_1 \\ \alpha_2 \\ \\ \alpha_r \end{matrix} & \begin{matrix} j, & 0, & \ldots & 0 \\ 0, & j, & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots & j \end{matrix} \end{array}, \qquad J_n = \begin{array}{c|c|} & \beta_1 \;\; \beta_2 \;\;\;\; \beta_s \\ \begin{matrix} \beta_1 \\ \beta_2 \\ \\ \beta_s \end{matrix} & \begin{matrix} j, & 0, & \ldots & 0 \\ 0, & j, & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots & j \end{matrix} \end{array}, \tag{A}$$

which are such that

$$J_m J_m = [1]_m^m, \quad J_n J_n = [1]_n^n, \tag{A$_1$}$$

will be called the *reversants*, or with greater definiteness the *part-reversants* of the respective class

$$M\begin{pmatrix} \alpha_1, \alpha_2, \ldots \alpha_r \\ \alpha_1, \alpha_2, \ldots \alpha_r \end{pmatrix}, \quad M\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ \beta_1, \beta_2, \ldots \beta_s \end{pmatrix}. \tag{3}$$

When J_m is applied as a prefactor to X, it reverses the order of arrangement of the horizontal rows in every constituent of X, leaving the class of X unaltered; and when J_n is applied as a postfactor to X, it reverses the order of arrangement of the vertical rows in every constituent of X, leaving the class of X unaltered. Thus we have

$$J_m X J_n = X', \; J_m X = Y, \; X J_n = Y', \tag{A$_2$}$$

where X', Y, Y' are the compound matrices of the same class as X

which are obtained by reversing the orders of arrangement of both the horizontal and vertical rows, the order of arrangement of the horizontal rows only, the order of arrangement of the vertical rows only in every one of the constituents of X. We will call X, X', Y, Y' four matrices of the class (2) *correlated by part-reversants.* If we denote these four matrices taken in any order by $X_1 X_2 X_3 X_4$, we have

$$X_k = A X_h B, \tag{A$_3$}$$

where A is $[1]_m^m$ or J_m according as the horizontal rows of corresponding constituents of X_h and X_k have the same or reversed orders of arrangement, and where B is $[1]_n^n$ or J_n according as the vertical rows of corresponding constituents of X_h and X_k have the same or reversed orders of arrangement. We can choose such an arrangement ξ_1, ξ_2, ξ_3, ξ_4 of the symbolic commutants (B) of Art. **2** that the equations (A$_3$) become the equations (C) of Art. **2** when X_1, X_2, X_3, X_4, are replaced respectively by ξ_1, ξ_2, ξ_3, ξ_4, and each of the part-reversants J_m and J_n is replaced by the symbolic reversant j; and we can then distinguish between the four correlated matrices X_1, X_2, X_3, X_4 by ascribing to them the types ξ_1, ξ_2, ξ_3, ξ_4. We can ascribe to any one of the four correlated matrices the type represented by any one of the four symbolic commutants; and the types of the other three correlated matrices are then uniquely determinate. The argument is exactly the same as in Art. **2**.3, the part-reversants J_m and J_n taking the place of the simple reversants J_m and J_n.

2. *Class-reversants.*

The compound square matrices

$$J_m = \begin{array}{c|cccc|} & \alpha_1 & \alpha_2 & & \alpha_r \\ \alpha_r & 0, & 0, & \dots & 1 \\ & \dots & \dots & \dots & \dots \\ \alpha_2 & 0, & 1, & \dots & 0 \\ \alpha_1 & 1, & 0, & \dots & 0 \end{array}, \qquad J_n = \begin{array}{c|cccc|} & \beta_s & & \beta_2 & \beta_1 \\ \beta_1 & 0, & \dots & 0, & 1 \\ \beta_2 & 0, & \dots & 1, & 0 \\ & \dots & \dots & \dots & \dots \\ \beta_s & 1, & \dots & 0, & 0 \end{array} \tag{B}$$

will be called the *class-reversants* of the respective classes

$$M\begin{pmatrix} \alpha_1, & \alpha_2, & \dots & \alpha_r \\ \alpha_r, & \dots & \alpha_2, & \alpha_1 \end{pmatrix}, \qquad M\begin{pmatrix} \beta_s & \dots & \beta_2, & \beta_1 \\ \beta_1, & \beta_2, & \dots & \beta_s \end{pmatrix}. \tag{4}$$

Their conjugates J_m' and J_n' are the class-reversants of the respective conjugate classes

$$M\begin{pmatrix} \alpha_r, & \dots & \alpha_2, & \alpha_1 \\ \alpha_1, & \alpha_2, & \dots & \alpha_r \end{pmatrix}, \qquad M\begin{pmatrix} \beta_1, & \beta_2, & \dots & \beta_s \\ \beta_s, & \dots & \beta_2, & \beta_1 \end{pmatrix} \tag{4$'$}$$

and are also the inverses of J_m and J_n; for we have

$$J_m J_m' = J_m' J_m = [1]_m^m, \quad J_n J_n' = J_n' J_n = [1]_n^n. \tag{B$_1$}$$

When J_m is applied as a prefactor to the compound matrix $X=[x]_m^n$ of the class (2), it reverses the order of arrangement of the horizontal minors of X without altering those minors themselves; and when J_n is applied as a postfactor to X, it reverses the order of arrangement of the vertical minors of X without altering those minors themselves. Thus we have

$$J_m X J_n = X', \quad J_m X = Y, \quad X J_n = Y', \tag{B$_2$}$$

where X', Y, Y' are the compound matrices of the respective classes

$$M\begin{pmatrix}\beta_s, \dots \beta_2, \beta_1 \\ \alpha_r, \dots \alpha_2, \alpha_1\end{pmatrix}, \quad M\begin{pmatrix}\beta_1, \beta_2, \dots \beta_s \\ \alpha_r, \dots \alpha_2, \alpha_1\end{pmatrix}, \quad M\begin{pmatrix}\beta_s, \dots \beta_2, \beta_1 \\ \alpha_1, \alpha_2, \dots \alpha_r\end{pmatrix} \tag{2$'$}$$

which are obtained when we reverse the orders of arrangement of the horizontal and vertical minors of X, the order of arrangement of the horizontal minors of X only, the order of arrangement of the vertical minors of X only, leaving the constituents of X unaltered in every case. We will call X, X', Y, Y' four matrices of the classes (2) and (2′) *correlated by class-reversants.* If we denote these four matrices taken in any order by X_1, X_2, X_3, X_4, it follows from the relations (B$_1$) that we have

$$X_k = A X_h B, \tag{B$_3$}$$

where A is either $[1]_m^m$ or one of the class-reversants J_m and J_m' according as the horizontal minors of X_h and X_k have the same or reversed orders of arrangement, and where B is either $[1]_n^n$ or one of the class-reversants J_n and J_n' according as the vertical minors of X_h and X_k have the same or reversed orders of arrangement, A and B being uniquely determinate when X_h and X_k are given. When A is $[1]_m^m$, we will regard it as being absent; and when B is $[1]_n^n$, we will regard it as being absent. We can choose such an arrangement ξ_1, ξ_2, ξ_3, ξ_4 of the symbolic commutants (B) of Art. 2 that the equations (A$_3$) become the equations (C) of Art. 2 when X_1, X_2, X_3, X_4 are replaced respectively by ξ_1, ξ_2, ξ_3, ξ_4, and the class reversants J_m, J_m', J_n, J_n' are replaced by the symbolic reversant j; and we can then distinguish between the four correlated matrices X_1, X_2, X_3, X_4 by ascribing to them the types

$\xi_1, \xi_2, \xi_3, \xi_4$. We can ascribe to any one of the four correlated matrices the type represented by any one of the four symbolic commutants; and the types of the other three correlated matrices are then uniquely determinate. The argument is the same as in Art. **2.3**, the class-reversants J_m and J_m' taking the place of the simple reversant J_m, and the class-reversants J_n and J_n' taking the place of the simple reversant J_n.

Ex. i. If we denote the part-reversants (A) by J_m and J_n, and the class-reversants (B) by K_m and K_n, we have

$$K_m J_m = |\,j\,|_m^m, \quad J_n K_n = [\,j\,]_n^n.$$

Consequently the successive applications of the prefactors J_m and K_m to the compound matrix $X = |\,x\,|_m^n$ of the class (2) simply reverse the order of arrangement of the horizontal rows of X; and the successive application of the postfactors J_n and K_n to X simply reverse the order of arrangement of the vertical rows of X.

3. *Commutantal products of compound matrices.*

A product of any number of compound matrices of specified classes will be called a *commutantal product* (in the symbolic sense) when:

(1) in every pair of consecutive factors the successive vertical index numbers of the factor on the left are the same as the successive horizontal index numbers of the factor on the right;

(2) such commutantal types are ascribed to the successive factor matrices that the product of the symbolic commutants representing their types is commutantal.

The product matrix in a commutantal product will always be regarded as belonging to the class defined by the successive horizontal index numbers of the first factor matrix and the successive vertical index numbers of the last factor matrix.

When correlated compound matrices and their types are defined as in sub-article 2 or 1 by class-reversants or part-reversants, the product matrices in any number of correlated commutantal products are correlated matrices to which the types of the corresponding commutantal products of symbolic commutants can be ascribed; and when any one of the product matrices is given, all the others are known by their types only.

For let $$[a]_m^p\,[b]_p^q\,[c]_q^r \ldots [k]_v^w\,[\,l\,]_w^n = [x]_m^n \quad \text{or} \quad ABC \ldots KL = X \qquad (5)$$

be any commutantal product of compound matrices in which the product of the symbolic commutants representing the types ascribed to $A, B, C, \dots K, L$ is

$$\alpha\beta\gamma \dots \kappa\lambda = \xi, \tag{6}$$

where ξ is a known symbolic commutant;

also let $[a']^p_m [b']^q_p [c']^r_q \dots [k']^w_v [l']^n_w = [x']^n_m$ or $A'B'C' \dots K'L' = X'$ (5')

be a correlated commutantal product derived from (5) by applying the appropriate class-reversants (or part-reversants) to some of the factors $A, B, \dots L$, and let the product of the known symbolic commutants representing the types of $A', B', C', \dots K', L'$ be

$$\alpha'\beta'\gamma' \dots \kappa'\lambda' = \xi' \tag{6'}$$

where ξ' is a known symbolic commutant;
further let J_m be the prefactor which reverses the order of arrangement of the horizontal minors of A (or the orders of arrangement of the horizontal rows of the successive horizontal minors of A), and let J_n be the postfactor which reverses the order of arrangement of the vertical minors of L (or the orders of arrangement of the vertical rows of the successive vertical minors of L). Then we have

$$X' = X, \text{ or } X' = J_m X J_n, \text{ or } X' = J_m X, \text{ or } X' = X J_n$$

according as

$$\xi' = \xi, \text{ or } \xi' = j\xi j, \text{ or } \xi' = j\xi, \text{ or } \xi' = \xi j.$$

In particular if we ascribe the type ξ to the product matrix X in (5), then the product matrix X' in (5') is the matrix of type ξ' correlated with X by class-reversants (or part-reversants).

The argument is the same as in Art. 2.4 when we replace each simple reversant by the appropriate class-reversant (or part-reversant).

We call (5) a *commutantal equation* of the *type* (6) when the product on the left in (5) is commutantal, and the types ascribed to $A, B, C \dots K, L$ and X are $\alpha, \beta, \gamma, \dots \kappa, \lambda$ and ξ. A commutantal equation of the form

$$HXK = Y$$

is called a *commutantal transformation* converting X into Y.

Ex. ii. The product $[a]^r_m [b]^s_r [c]^t_s [d]^n_t = [x]^n_m$ or $ABCD = X$ (7).

in which A, B, C, D, X are compound matrices having the class-symbols

$$\begin{pmatrix} u_1, u_2, \dots u_\rho \\ m_1, m_2, \dots m_\mu \end{pmatrix}, \begin{pmatrix} v_1, v_2, \dots v_\sigma \\ u_1, u_2, \dots u_\rho \end{pmatrix}, \begin{pmatrix} w_1, w_2, \dots w_\tau \\ v_1, v_2 \dots v_\sigma \end{pmatrix}, \begin{pmatrix} n_1, n_2, \dots n_\nu \\ w_1, w_2, \dots w_\tau \end{pmatrix}, \begin{pmatrix} n_1, n_2, \dots n_\nu \\ m_1, m_2, \dots m_\mu \end{pmatrix}$$

is commutantal when we ascribe to A, B, C, D the types

$$\alpha=\{\pi',\pi\},\ \beta=\{\pi,\pi'\},\ \gamma=\{\pi',\pi'\},\ \delta=\{\pi',\pi\},$$

because then $\alpha\beta\gamma\delta$ is the commutantal product

$$\alpha\beta\gamma\delta=\{\pi',\pi\}\{\pi,\pi'\}\{\pi',\pi'\}\{\pi',\pi\}=\xi,\ \text{where}\ \xi=\{\pi',\pi\}. \tag{8}$$

Ex. iii. Let $$[a']^{r}_{m}[b']^{s}_{r}[c']^{t}_{s}[d']^{n}_{t}=[x']^{n}_{m} \quad \text{or} \quad A'B'C'D'=X' \tag{7'}$$

be the commutantal product correlated with (7) by class-reversants in which the types of A', B', C', D' are

$$\alpha'=\{\pi',\pi'\},\ \beta'=\{\pi',\pi'\},\ \gamma'=\{\pi'\ \pi\},\ \delta'=\{\pi,\pi'\},$$

so that $\alpha'\beta'\gamma'\delta'$ is the commutantal product

$$\alpha'\beta'\gamma'\delta'=\{\pi',\pi'\}\{\pi',\pi'\}\{\pi',\pi\}\{\pi,\pi'\}=\xi',\ \text{where}\ \xi'=\{\pi',\pi'\}. \tag{8'}$$

Then if J_r, J_s, J_n are the class-reversants of the respective classes

$$M\begin{pmatrix}u_\rho, \ldots u_2, u_1\\ u_1, u_2, \ldots u_\rho\end{pmatrix},\ M\begin{pmatrix}v_\sigma, \ldots v_2, v_1\\ v_1, v_2, \ldots v_\sigma\end{pmatrix},\ M\begin{pmatrix}n_\nu \ldots n_2, n_1\\ n_1, n_2, \ldots n_\nu\end{pmatrix},$$

and if J_r', J_s' are the conjugates and inverses of J_r, J_s, we have

$$X'=AJ_r\,.\,J_r'B\,.\,CJ_s\,.\,J_s'DJ_n=ABCD\,.\,J_n=XJ_n,$$

$$\xi'=\alpha j\,.\,j\beta\,.\,\gamma j\,.\,j\delta j, \qquad =\alpha\beta\gamma\delta\,.\,j \qquad =\xi j,$$

the second equation being representative of the first.

The relation of X' to X follows from the known values of ξ and ξ'; for the equation $\xi'=\xi j$ shows that $X'=XJ_n$, i.e. X' is the matrix obtained by reversing the order of arrangement of the vertical minors of X. If we ascribe to X the type $\xi=\{\pi',\pi\}$, then X' is the matrix of type $\xi'=\{\pi',\pi'\}$ correlated with X by class-reversants.

Ex. iv. The class-symbols of A', B', C', D', X' in (7′) are

$$\begin{pmatrix}u_\rho, \ldots u_2, u_1\\ m_1, m_2, \ldots m_\mu\end{pmatrix},\ \begin{pmatrix}v_1, v_2, \ldots v_\sigma\\ u_\rho, \ldots u_2, u_1\end{pmatrix},\ \begin{pmatrix}w_\tau, \ldots w_2, w_1\\ v_1, v_2, \ldots v_\sigma\end{pmatrix},\ \begin{pmatrix}n_\nu, \ldots n_2, n_1\\ w_\tau, \ldots w_2, w_1\end{pmatrix},\ \begin{pmatrix}n_\nu, \ldots n_2, n_1\\ m_1, m_2, \ldots m_\mu\end{pmatrix}.$$

If (7′) were correlated with (7) by part-reversants, the class-symbols A', B', C', D', X' would be the same as those of A, B, C, D, X in (7).

4. Hemipteric matrices.

1. *The four types of hemipteric matrices.*

A compound matrix X whose successive horizontal and vertical *index numbers* are $\alpha_1, \alpha_2, \ldots \alpha_r$ and $\beta_1, \beta_2, \ldots \beta_s$ will be called a ***hemipteric matrix*** or a *hemipteric* of the class

$$M\begin{pmatrix}\beta_1, \beta_2, \ldots \beta_s\\ \alpha_1, \alpha_2, \ldots \alpha_r\end{pmatrix} \tag{A}$$

in the four cases when every constituent which lies below (to the left of) or above (to the right of) either diagonal, or above (to the left of) or below (to the right of) either counter-diagonal is a zero matrix. We can distinguish between these four kinds of hemipteric matrices by ascribing to them the respective commutantal types

$$\{\pi, \pi\}, \{\pi', \pi'\}, \{\pi', \pi\}, \{\pi, \pi'\}; \tag{B}$$

and we will define the hemipteric matrices of the class (A) to be:

(1) those of the 1st type or of the type $\{\pi, \pi\}$ in which every constituent lying below (or to the left of) either of the two diagonals is a zero matrix;

(2) those of the 2nd type or of the type $\{\pi', \pi'\}$ in which every constituent lying above (or to the right of) either of the two diagonals is a zero matrix;

(3) those of the 3rd type or of the type $\{\pi', \pi\}$ in which every constituent lying above (or to the left of) either of the two counter-diagonals is a zero matrix;

(4) those of the 4th type or of the type $\{\pi, \pi'\}$ in which every constituent lying below (or to the right of) either of the two counter-diagonals is a zero matrix.

The term 'hemipteric' is most appropriate when the horizontal and vertical minors are equal in number, i.e. when $s = r$, so that the two diagonals coincide, and the two counter-diagonals coincide. A hemipteric matrix in which this condition is satisfied will be called a *pure hemipteric matrix*. When $s = r$, the constituents of a compound matrix X of the class (A) can be regarded as forming a diagonal (or counter-diagonal) and two wings, one wing lying on each side of the diagonal (or counter-diagonal); and X becomes hemipteric when one of the two wings is made to vanish.

The structures of hemipteric matrices are shown in Exs. i, ii and iii, where the constituents represented by 0's are zero matrices. In *general hemipteric matrices* the other constituents, represented by the letter X with suffixes, are matrices whose elements are all arbitrary; particular hemipteric matrices are obtained by giving particular values to the elements of those arbitrary constituents.

Ex. i. If $s = r$, the standard skeletons of general hemipteric matrices X of the class (A) and of the respective types (B) will be taken to be the matrices

$$\begin{bmatrix} X_{11} & X_{12} & \dots & X_{1r} \\ 0 & X_{22} & \dots & X_{2r} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X_{rr} \end{bmatrix}, \quad \begin{bmatrix} X_{rr} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ X_{2r} & \dots & X_{22} & 0 \\ X_{1r} & \dots & X_{12} & X_{11} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \dots & X_{rr} \\ \dots & \dots & \dots & \dots \\ 0 & X_{22} & \dots & X_{2r} \\ X_{11} & X_{12} & \dots & X_{1r} \end{bmatrix}, \quad \begin{bmatrix} X_{1r} & \dots & X_{12} & X_{11} \\ X_{2r} & \dots & X_{22} & 0 \\ \dots & \dots & \dots & \dots \\ X_{rr} & \dots & 0 & 0 \end{bmatrix} \quad \dots (A_1)$$

which are correlated by simple reversals. These are the standard skeletons of general *pure hemipteric matrices* denoted by X.

Ex. ii. If $s > r$, the standard skeletons of general hemipteric matrices X of the class (A) and of the respective types (B) will be taken to be the matrices

$$\begin{array}{ccccc} 0 \ldots & X_{11} & X_{12} & \ldots & X_{1r} \\ 0 \ldots & 0 & X_{22} & \ldots & X_{2r} \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots} \\ 0 \ldots & 0 & 0 & \ldots & X_{rr} \end{array}, \quad \left|\begin{array}{ccccc} X_{rr} & \ldots & 0 & 0 & \ldots 0 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots} \\ X_{2r} & \ldots & X_{22} & 0 & \ldots 0 \\ X_{1r} & \ldots & X_{12} & X_{11} & \ldots 0 \end{array}\right|, \quad \left|\begin{array}{ccccc} 0 \ldots & 0 & 0 & \ldots & X_{rr} \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots} \\ 0 \ldots & 0 & X_{22} & \ldots & X_{2r} \\ 0 \ldots & X_{11} & X_{12} & \ldots & X_{1r} \end{array}\right|,$$

$$\left|\begin{array}{ccccc} X_{1r} & \ldots & X_{12} & X_{11} & \ldots 0 \\ X_{2r} & \ldots & X_{22} & 0 & \ldots 0 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots} \\ X_{rr} & \ldots & 0 & 0 & \ldots 0 \end{array}\right| \qquad \ldots (A_2)$$

which are correlated by simple reversals. In each case X has $s-r$ initial or final *zero vertical minors* which are represented by $s-r$ initial or final vertical rows of 0's in the skeleton. The pure hemipteric matrix which remains when the zero vertical minors are struck out will be called the *general pure part* of X.

Ex. iii. If $r > s$, the standard skeletons of general hemipteric matrices X of the class (A) and of the respective types (B) will be taken to be the matrices

$$\left|\begin{array}{cccc} X_{11} & X_{12} & \ldots & X_{1s} \\ 0 & X_{22} & \ldots & X_{2s} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & \ldots & X_{ss} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & \ldots & 0 \end{array}\right|, \quad \left|\begin{array}{cccc} 0 & \ldots & 0 & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ X_{ss} & \ldots & 0 & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ X_{2s} & \ldots & X_{22} & 0 \\ X_{1s} & \ldots & X_{12} & X_{11} \end{array}\right|, \quad \left|\begin{array}{cccc} 0 & 0 & \ldots & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & \ldots & X_{ss} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ 0 & X_{22} & \ldots & X_{2s} \\ X_{11} & X_{12} & \ldots & X_{1s} \end{array}\right|, \quad \left|\begin{array}{cccc} X_{1s} & \ldots & X_{12} & X_{11} \\ X_{2s} & \ldots & X_{22} & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ X_{ss} & \ldots & 0 & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ 0 & \ldots & 0 & 0 \end{array}\right|$$

$$(A_3)$$

which are correlated by simple reversals. In each case X has $r-s$ final or initial *zero horizontal minors* which are represented by $r-s$ final or initial horizontal rows of 0's in the skeleton. The pure hemipteric matrix which remains when the zero horizontal minors are struck out will be called the *general pure part of* X.

Ex. iv. All particular hemipteric matrices X of the respective types (B) which are not zero matrices can be expressed in the corresponding forms

$$\begin{bmatrix} 0, & x' \\ 0, & 0 \end{bmatrix}_{\mu,\, p}^{q,\, \nu}, \quad \begin{bmatrix} 0, & 0 \\ x', & 0 \end{bmatrix}_{p,\, \mu}^{\nu,\, q}, \quad \begin{bmatrix} 0, & 0 \\ 0, & x' \end{bmatrix}_{p,\, \mu}^{q,\, \nu}, \quad \begin{bmatrix} x', & 0 \\ 0, & 0 \end{bmatrix}_{\mu,\, p}^{\nu,\, q}, \qquad \ldots (A_4)$$

where in each case the minor $X' = [x']_{\mu}^{\nu}$ is a pure hemipteric matrix of the same type as X in which the constituents of the leading diagonal (for the first two types) or of the counter-diagonal (for the last two types) are not all zero matrices; and where in each case μ is the sum of a certain number ρ of the initial or final horizontal index numbers, ν is the sum of an equal number ρ of the final or initial vertical index numbers, p is the sum of the remaining horizontal index numbers, and q is the sum of the remaining vertical index numbers. When X is a general hemipteric matrix, either p or q is 0.

The pure hemipteric matrix X' will be called the *effective pure part* of X. The number ρ, which is the number of horizontal minors and also the number of the vertical minors of X', will be called the *parametric* rank of X. The number of final or initial zero horizontal minors in X the sum of whose horizontal orders is p will be called the *horizontal nullity* of X; and the number of initial or final zero vertical minors in X the sum of whose vertical orders is q will be called the *vertical nullity* of X. A hemipteric matrix such as X' whose horizontal and vertical nullities are both equal to 0 will be called *parametrically undegenerate*. Every such hemipteric matrix is necessarily pure.

2. *Uses of the terms 'ante-hemipteric,' 'counter-hemipteric' and 'parametric' as applied to hemipteric matrices.*

A hemipteric matrix will be called *ante-hemipteric* when it has one of the first two types $\{\pi, \pi\}$, $\{\pi', \pi'\}$, and *counter-hemipteric* when it has one of the last two types $\{\pi', \pi\}$, $\{\pi, \pi'\}$.

The *parametric constituents* of a general hemipteric matrix X are those constituents whose elements are arbitrary parameters, i.e. those constituents which are not zero matrices. They are identical with the parametric constituents of the general pure part of X as defined in Exs. ii and iii.

Let X be a general hemipteric matrix having the standard skeleton appropriate to its type as described in Exs. i, ii and iii, so that the same notations are used for the parametric constituents of X as for the parametric constituents of the pure part of X. Then the parametric constituent X_{ij} (in which $j \nless i$) will be said to have *difference-weight* $j-i$. The parametric constituents of difference-weight k form the $(k+1)$th *parametric diagonal* line of X; in particular the parametric constituents X_{ii} of difference-weight 0 form the *first-parametric diagonal line* of X, which will also be called the *parametric diagonal* of X. The parametric diagonal lines of X are diagonal lines or counter-diagonal lines according as X is ante-hemipteric or counter-hemipteric; in particular the parametric diagonal of X is the diagonal or counter-diagonal of the general pure part of X according as X is ante-hemipteric or counter-hemipteric. The constituents of difference-weight 0 forming the parametric diagonal of X are the *parametric diagonal constituents*; these are counted from X_{11} as the first, X_{ii} being the ith parametric diagonal constituent. A *parametric diagonal super-minor* of X is a super-minor formed by the intersections of the horizontal and vertical rows of a number of parametric diagonal constituents, and it is *unbroken* when it is corranged and those constituents are consecutive. The horizontal and vertical rows of X passing through the ith parametric diagonal constituent X_{ii} form respectively the ith *parametric horizontal minor* and the ith *parametric vertical minor* of X.

It will be observed that the qualification 'parametric' serves the three purposes of indicating the presence of arbitrary constituents, rendering terms applicable to pure hemipteric matrices of the first type also applicable to all hemipteric matrices, and indicating a definite order of counting depending on the type of the hemipteric matrix.

If the general hemipteric matrix X belongs to a class (A) in which $s>r$, it contains $s-r$ zero vertical minors which precede the parametric vertical minors, and these are called the *pre-parametric vertical minors*; if X belongs to a class (A) in which $r>s$, it contains $r-s$ zero horizontal minors which follow the parametric horizontal minors, and these are called the *post-parametric horizontal minors.* The pre-parametric and parametric vertical minors taken together and counted in the same order of succession as the parametric vertical minors will be called the *major-parametric vertical* minors; and the parametric and post-parametric horizontal minors taken together and counted in the same order of succession as the parametric horizontal minors will be called the *major-parametric horizontal minors.* In a hemipteric matrix of the first type the ith major-parametric horizontal or vertical minor is actually the ith horizontal or vertical minor.

All the above terms will be applied to a particular hemipteric matrix X when we regard X as a general hemipteric matrix whose parametric constituents have been particularised.

3. *Apical and basical constituents and super-minors; median lines; quadrate and quasi-scalaric hemipteric matrices.*

We will define the *apex* of a hemipteric matrix X to be

the right-hand top corner	when the type is $\{\pi, \pi\}$,
the left-hand bottom corner	when the type is $\{\pi', \pi'\}$,
the right-hand bottom corner	when the type is $\{\pi', \pi\}$,
the left-hand top corner	when the type is $\{\pi, \pi'\}$;

and the *base* to be the corner opposite to the vertex. The *apical constituent* is the parametric constituent at the apex, and the *basical constituent* is the zero constituent at the base. An *apical super-minor* or a *basical super-minor* is an unbroken corranged super-minor which contains the apical or the basical constituent. The *median line* of X is that counter-diagonal (when X is ante-hemipteric) or that diagonal (when X is counter-hemipteric) which contains the apical constituent.

When one of the types (B) is ascribed to any compound matrix X of a specified class, the terms 'apical,' 'basical,' 'median' as well as the term 'parametric' will be used with reference to X in the same senses as if X were hemipteric.

A quadrate hemipteric matrix will be defined to be a pure hemipteric matrix in which the constituents of the parametric diagonal are all square matrices. Such a matrix is necessarily square; in fact the class-symbols of a quadrate ante-hemipteric matrix and a quadrate counter-hemipteric matrix have the respective forms

$$\begin{pmatrix} a_1, a_2, \ldots a_r \\ a_1, a_2, \ldots a_r \end{pmatrix}, \quad \begin{pmatrix} a_1, a_2, \ldots a_r \\ a_r, \ldots a_2, a_1 \end{pmatrix}.$$

A *quasi-scalaric hemipteric matrix* is a quadrate hemipteric matrix in which all constituents except those forming the parametric diagonal are zero matrices. Each of the parametric diagonal constituents may be either a non-zero matrix or a zero matrix.

4. *Correlated hemipteric matrices.*

The four matrices correlated by class-reversants which can be derived from any one given hemipteric matrix are when suitably arranged four hemipteric matrices X, X', Y, Y' of the respective types

$$\xi=\{\pi, \pi\}, \quad \xi'=\{\pi', \pi'\}, \quad \eta=\{\pi', \pi\}, \quad \eta'=\{\pi, \pi'\} \tag{1}$$

which are *formed with the same constituents*, their skeletons (for which the standard notations of Exs. i, ii and iii can be used) being correlated by simple reversants. The types of X, X', Y, Y' are fixed by the definitions of the types of hemipteric matrices, but are in accordance with Ex. iii of Art. 2.

In fact let $X=[x]_m^n$ be a hemipteric matrix of the type $\{\pi, \pi\}$ and of a class (A) in which

$$a_1+a_2+\ldots+a_r=m, \quad \beta_1+\beta_2+\ldots+\beta_s=n;$$

and let J_m, J_m', J_n, J_n' be the class-reversants of the classes

$$M\begin{pmatrix} a_r, \ldots a_2, a_1 \\ a_1, a_2, \ldots a_r \end{pmatrix}, \quad M\begin{pmatrix} a_1, a_2, \ldots a_r \\ a_r, \ldots a_2, a_1 \end{pmatrix}, \quad M\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ \beta_s, \beta_2, \ldots \beta_1 \end{pmatrix}, \quad M\begin{pmatrix} \beta_s, \ldots \beta_2, \beta_1 \\ \beta_1, \beta_2, \ldots \beta_s \end{pmatrix}. \tag{2}$$

Then if X', Y, Y' are the three other matrices which can be derived from X by class-reversants, these being suitably arranged, we have

$$\begin{aligned} X' &= J_m X J_n, & Y &= J_m X, & Y' &= X J_n, \\ X &= J_m' X' J_n', & Y' &= J_m' X', & Y &= X' J_n, \\ Y' &= J_m' Y J_n, & X &= J_m' Y, & X' &= Y J_n, \\ Y &= J_m Y' J_n', & X' &= J_m Y', & X &= Y' J_n; \end{aligned} \tag{3}$$

and X, X', Y, Y' are four hemipteric matrices of the respective types (1) and of the respective classes

$$M\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ a_1, a_2, \ldots a_r \end{pmatrix}, \quad M\begin{pmatrix} \beta_s, \ldots \beta_2, \beta_1 \\ a_r, \ldots a_2, a_1 \end{pmatrix}, \quad M\begin{pmatrix} \beta_s, \ldots \beta_2, \beta_1 \\ a_1, a_2, \ldots a_r \end{pmatrix}, \quad M\begin{pmatrix} \beta_1, \beta_2, \ldots \beta_s \\ a_r, \ldots a_2, a_1 \end{pmatrix} \tag{3'}$$

which have the properties just described. So far as types are concerned, we can regard X, X', Y, Y' as replaceable by the symbolic commutants ξ, ξ', η, η', the class-reversants J_m, J_m', J_n, J_n' being simultaneously replaceable by the symbolic commutant j, and the relations (3) being replaceable by the relations (B_2) of Art. 2.

The four matrices correlated by simple reversants which can be derived from any one given hemipteric matrix are also four hemipteric matrices of the same four types and the same four classes as above, but their constituents are not the same, corresponding constituents being correlated by simple reversants.

5. *Commutantal products of hemipteric matrices.*

A product of any number of hemipteric matrices of given classes is a *commutantal product* (in the symbolic sense) when :

(1) in every pair of consecutive factors the successive vertical index numbers of the factor on the left are the same as the successive horizontal index numbers of the factor on the right ;

(2) the product of the symbolic commutants representing the types of the successive factor matrices is commutantal.

The definition is the same as that of a commutantal product of compound matrices except that now the types of the factor matrices are fixed.

It is easily seen by evaluating a commutantal product of two general hemipteric matrices of the first type that :

In every commutantal product of hemipteric matrices of the first type the product matrix is another hemipteric matrix of the first type belonging to the class defined by the successive horizontal index numbers of the first factor matrix and the successive vertical index numbers of the last factor matrix.

Let $$ABC \dots KL = X \tag{4}$$
and $$\alpha\beta\gamma \dots \kappa\lambda = \xi \tag{5}$$

be respectively any commutantal product whatever of hemipteric matrices, and the product of the symbolic commutants representing the types of the factor matrices A, B, C, ... K, L ; and let

$$A'B'C' \dots K'L' = X' \tag{4'}$$
and $$\alpha'\beta'\gamma' \dots \kappa'\lambda' = \xi' \tag{5'}$$

be respectively any correlated commutantal product of hemipteric matrices derived from (4) by applying the appropriate class-reversants to some of the factor matrices A, B, ... L, and the product of the symbolic commutants representing the types of the new factor matrices A', B', C', ... K', L'. We know that if the type ξ' is ascribed to X', then X is the matrix of type ξ correlated with X' by class-reversants. Now we can choose A', B', ... L' to be all of the first type; then we have $\xi' = \{\pi, \pi\}$. from (4′) it follows that X' has the type ξ', and we conclude that X has the type ξ.

Thus in every commutantal product of hemipteric matrices the product matrix is another hemipteric matrix of the type determined by the product of the symbolic commutants representing the types of the successive factor matrices, and of the class defined by the successive horizontal index numbers of the first factor matrix and the successive vertical index numbers of the last factor matrix.

Further in any number of correlated commutantal products of hemipteric matrices such as (4) and (4′) the product matrices are correlated hemipteric matrices, and they are all known by their types only when any one of them is given. In fact the expressions for the parametric constituents and elements of the product matrix in terms of the parametric constituents and elements of the factor matrices are the same for every one of the products.

6. *The conjugate of any hemipteric matrix; the inverse and conjugate reciprocal of a quadrate hemipteric matrix.*

If X is a hemipteric matrix of the class $M\begin{pmatrix}\beta_1, \beta_2, \ldots \beta_s \\ \alpha_1, \alpha_2, \ldots \alpha_r\end{pmatrix}$ having any one of the four types

$$\{\pi, \pi\},\ \{\pi', \pi'\},\ \{\pi', \pi\},\ \{\pi, \pi'\}, \tag{6}$$

its *conjugate* (or skew-conjugate) is a hemipteric matrix of the conjugate class $M\begin{pmatrix}\alpha_1, \alpha_2, \ldots \alpha_r \\ \beta_1, \beta_2, \ldots \beta_s\end{pmatrix}$ having the correspondingly situated type in the series

$$\{\pi', \pi'\},\ \{\pi, \pi\},\ \{\pi', \pi\},\ \{\pi, \pi'\}. \tag{7}$$

A quadrate hemipteric matrix is undegenerate when and only when all its parametric diagonal constituents are undegenerate. The *inverse* of an undegenerate quadrate hemipteric matrix X having one of the four types (6) is an undegenerate quadrate hemipteric matrix X^{-1} having the correspondingly situated type in the series

$$\{\pi, \pi\},\ \{\pi', \pi'\},\ \{\pi, \pi'\},\ \pi', \pi\}, \tag{8}$$

the classes of X and X^{-1} being representable by

$$M\begin{pmatrix} a_1, a_2, \dots a_r \\ a_1, a_2, \dots a_r \end{pmatrix} \text{ and } M\begin{pmatrix} a_1, a_2, \dots a_r \\ a_1, a_2, \dots a_r \end{pmatrix} \text{ when } X \text{ is ante-hemipteric,}$$

$$M\begin{pmatrix} a_1, a_2, \dots a_r \\ a_r, \dots a_2, a_1 \end{pmatrix} \text{ and } M\begin{pmatrix} a_r, \dots a_2, a_1 \\ a_1, a_2, \dots a_r \end{pmatrix} \text{ when } X \text{ is counter-hemipteric.}$$

It is sufficient to consider the case in which X has the type $\{\pi, \pi\}$, the other three cases being deducible by the properties of active and passive minors.

The *conjugate reciprocal* of any quadrate hemipteric matrix X is a quadrate hemipteric matrix having the same class and type as those just described for X^{-1} in the special case when X is undegenerate.

Corresponding situated types in the series (6) and (7) will be called *mutually conjugate types*; and correspondingly situated types in the series (6) and (8) will be called *mutually inverse types*. Two mutually conjugate or two mutually inverse types are always either both ante-types or both counter-types. An ante-type is inverse to itself but not conjugate to itself; a counter-type is conjugate to itself but not inverse to itself.

Ex. v. *Quadrate hemipteric matrices to which more than one type can be ascribed.*

Let X be a quadrate hemipteric matrix of type ξ belonging to a given class in which there are no zero index numbers. Then if X is undegenerate, it is possible to ascribe a second type to it when and only when it is quasi-scalaric, the second type being then either the conjugate or the inverse of ξ. If any other type can be ascribed to X, i.e. if X can be regarded as having both an ante-type and a counter-type, it must be degenerate; for either the first or the last parametric diagonal constituent must be a zero matrix.

Let X' be the conjugate of X having the type ξ' conjugate to ξ. Then if X and X' are undegenerate, it is possible to ascribe a second type to X' when and only when X and X' are quasi-scalaric, the second type being then the inverse of ξ. If any other type can be ascribed to X', then X and X' must be degenerate.

Let X be undegenerate, and let X^{-1} be the inverse of X having the type ξ' inverse to ξ. Then it is possible to ascribe a second type to X' when and only when X and X' are quasi-scalaric, the second type being then the conjugate of ξ.

Let X' be the conjugate reciprocal of X having the type ξ' inverse to ξ. Then if X and X' are undegenerate, it is possible to ascribe a second type to X' when and only when X and X' are quasi-scalaric, the second type being then the conjugate of ξ. If any other type can be ascribed to X', then X and X' must be degenerate.

7. *Symmetric and skew-symmetric hemipteric matrices.*

By a *symmetric class* of a compound matrix will be meant a class whose successive horizontal index numbers and successive vertical index numbers form the same sequence in which any two elements equidistant from the first and last are equal. Thus the general symbol for a symmetric class is

$$M\begin{pmatrix} a_1, a_2, \dots a_r \\ a_1, a_2, \dots a_r \end{pmatrix}, \text{ where } a_{r+1-i} = \tag{9}$$

and such a class is the same as the conjugate class.

From the fact that a symmetric or skew-symmetric matrix must be a square matrix in which the zero elements are symmetric with respect to the leading diagonal it follows that:

A counter-hemipteric matrix of either possible type which is symmetric or skew-symmetric must be expressible as a (quadrate) hemipteric matrix of the same type of some symmetric class when its successive index numbers are made as small as possible.

An ante-hemipteric matrix of either possible type which is symmetric or skew-symmetric must be expressible as a (quadrate) quasi-scalaric hemipteric matrix of the same type when its successive index numbers are made as small as possible.

Confining our attention now to quadrate hemipteric matrices, and expressing the conditions that a matrix X whose conjugate is X' shall be symmetric or skew-symmetric in the form

$$X = \epsilon X',$$

where $\epsilon = +1$ or $\epsilon = -1$ according as X is to be symmetric or skew-symmetric, we obtain the following two results.

A quadrate hemipteric matrix X of any given symmetric class having one of the counter-types $\{\pi', \pi\}$, $\{\pi, \pi'\}$ is symmetric (or skew-symmetric) if and only if it is symmetric (or skew-symmetric) with respect to the median line, the constituents of the median line itself being all symmetric (or all skew-symmetric).

If in this first case we use for X, whose class is symmetric, one of the standard skeletons

$$\begin{bmatrix} 0 & 0 & \dots & X_{rr} \\ \dots & \dots & \dots & \dots \\ 0 & X_{22} & \dots & X_{2r} \\ X_{11} & X_{12} & \dots & X_{1r} \end{bmatrix}, \quad \begin{vmatrix} X_{1r} & \dots & X_{12} & X_{11} \\ X_{2r} & \dots & X_{22} & 0 \\ \dots & \dots & \dots & \dots \\ X_{rr} & \dots & 0 & 0 \end{vmatrix}$$

and if X_{ij}', X_{vu}' are the conjugates of the constituents X_{ij}, X_{vu}, the necessary and sufficient conditions that X shall be symmetric or skew-symmetric are

$$X_{ij} = \epsilon X_{vu}' \text{ whenever } u+i=v+j=r+1,$$

where $\epsilon=+1$ or $\epsilon=-1$ according as X is to be symmetric or skew-symmetric. In particular for the constituents of the median line we must have

$$X_{ij} = \epsilon X_{ij}' \text{ whenever } i+j=r+1.$$

A quadrate hemipteric matrix X of any given class having one of the ante-types $\{\pi, \pi\}$, $\{\pi', \pi'\}$ cannot be symmetric or skew-symmetric unless it is quasi-scalaric; and when this condition is satisfied, it is symmetric (or skew-symmetric) if and only if its diagonal constituents are all symmetric (or all skew-symmetric).

If in this second case we use for X, whose class is quadrate, the standard skeleton appropriate to its type, the necessary and sufficient conditions that X shall be symmetric or skew-symmetric are

$$X_{ij}=0 \text{ when } j \neq i, \quad X_{ii} = \epsilon X_{ii}',$$

which, because $X_{ij}=0$ when $j<i$, are equivalent to $X_{ij}=\epsilon X_{ji}'$.

8. *Hemipteric class symbols.*

When a hemipteric matrix X of the first type $\{\pi, \pi\}$ and of the class

$$M\begin{pmatrix} \mu_1, \mu_2, \dots \mu_q, \ \beta_1, \beta_2, \dots \beta_r \\ \alpha_1, \alpha_2, \dots \alpha_r, \ \lambda_1, \lambda_2, \dots \lambda_p \end{pmatrix}$$

has p final horizontal and q initial vertical zero-minors and differs only by them from a pure hemipteric matrix Y of the same type, we may represent X and Y by the *hemipteric class-symbols*

$$X = \begin{pmatrix} \mu_1, \mu_2, \dots \mu_q, \ \beta_1, \beta_2, \dots \beta_r & \\ \quad \alpha_1, \alpha_2, \dots & \alpha_r, \lambda_1, \lambda_2, \dots \lambda_p \end{pmatrix}, \quad Y = \begin{pmatrix} \beta_1, \beta_2, \dots \beta_r \\ \alpha_1, \alpha_2, \dots \alpha_r \end{pmatrix}, \qquad (10)$$

the equations (10) standing for equations of the schematic form:

$$X = \begin{array}{c|ccccccc|} & \mu_1 & & \mu_q & \beta_1 & \beta_2 & & \beta_r \\ \alpha_1 & 0 & \dots & 0 & \times & \times & \dots & \times \\ \alpha_2 & 0 & \dots & 0 & 0 & \times & \dots & \times \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_r & 0 & \dots & 0 & 0 & 0 & \dots & \times \\ \lambda_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda_p & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}, \qquad Y = \begin{array}{c|cccc|} & \beta_1 & \beta_2 & & \beta_r \\ \alpha_1 & \times & \times & \dots & \times \\ \alpha_2 & 0 & \times & \dots & \times \\ & \dots & \dots & \dots & \dots \\ \alpha_r & 0 & 0 & \dots & \times \end{array}, \qquad (10')$$

where Y is the effective pure part of X. In these symbols of the type $\{\pi, \pi\}$ the *initial vertical* index numbers under which no horizontal index numbers occur represent corresponding initial vertical *zero minors*, and the *final horizontal* index numbers above which no vertical index numbers occur represent corresponding final horizontal zero minors. Thus in the symbol for X the vertical index number μ_j represents the zero vertical minor of vertical order μ_j, and the horizontal index number λ_i represents the zero horizontal minor of horizontal order λ_i.

The practical use of these symbols rests on the following properties of any pure hemipteric matrix of the type $\{\pi, \pi\}$.

(1) If we strike out q *initial horizontal* minors, we obtain a hemipteric matrix of the same type which contains q initial vertical zero minors.

(2) If we strike out p *final vertical* minors, we obtain a hemipteric matrix of the same type which contains p final horizontal zero minors.

Thus if Y_1 and Y_2 are the matrices formed respectively by striking out the first horizontal and the last vertical minors of Y, then Y_1 and Y_2 contain respectively one initial vertical and one final horizontal zero minor; and this is shown by their hemipteric class-symbols

$$Y_1 = \begin{pmatrix} \beta_1, & \beta_2, & \ldots & \beta_r \\ & \alpha_2, & \ldots & \alpha_r \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \beta_1, & \ldots & \beta_{r-1} & \\ \alpha_1, & \ldots & \alpha_{r-1}, & \alpha_r \end{pmatrix},$$

which are formed respectively from the class-symbol for Y by striking out the first horizontal index number and the last vertical index number. In the same way we could regard X as the super-minor of a pure hemipteric matrix of the class

$$M \begin{pmatrix} \mu_1, & \mu_2, & \ldots & \mu_q, & \beta_1, & \beta_2, & \ldots & \beta_r, & \sigma_1, & \sigma_2, & \ldots & \sigma_p \\ \rho_1, & \rho_2, & \ldots & \rho_q; & \alpha_1, & \alpha_2, & \ldots & \alpha_r, & \lambda_1, & \lambda_2, & \ldots & \lambda_p \end{pmatrix}$$

formed by striking out the first q horizontal minors (or horizontal index numbers) and the last p vertical minors (or vertical index numbers).

In using the hemipteric class-symbol for X, the constituents of X of orders $\begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_2 \\ \alpha_2 \end{pmatrix}, \ldots \begin{pmatrix} \beta_r \\ \alpha_r \end{pmatrix}$, which are the diagonal constituents of Y, will be called the *effective diagonal constituents* of X. If we were to fill up the vacant spaces in the symbol for X with 0's, we should convert it into an ordinary class-symbol in which the effective diagonal constituents have become actual diagonal constituents.

The hemipteric class-symbol for the hemipteric matrices X' and Y' of type $\{\pi', \pi'\}$ correlated with X and Y are

$$X' = \begin{pmatrix} & \beta_r, \dots \beta_2, \beta_1, & \mu_q, \dots \mu_2, \mu_1 \\ \lambda_p, \dots \lambda_2, \lambda_1, & \alpha_r, \dots \alpha_2, \alpha_1 & \end{pmatrix}, \quad Y' = \begin{pmatrix} \beta_r, \dots \beta_2, \beta_1 \\ \alpha_r, \dots \alpha_2, \alpha_1 \end{pmatrix}.$$

Hemipteric class-symbols of the types $\{\pi', \pi\}$, $\{\pi, \pi'\}$ are less simple to use, and are not needed.

Ex. vi. *Evaluation of any commutantal product of two hemipteric matrices of the first type.*

Let the parametric rank and the horizontal and vertical nullities of a hemipteric matrix be defined as in Ex. iv. of Art. 4; let

$$AB = X$$

be a commutantal product of two hemipteric matrices A and B of the first type whose parametric ranks are α and β, whose horizontal nullities are ξ and η, whose vertical nullities are λ and μ, and whose effective parts are A' and B'; and let

$$m = \alpha + \xi, \quad p = \alpha + \lambda = \beta + \eta, \quad n = \beta + \mu,$$

so that m is the total number of horizontal index numbers in A, p is the total number of vertical index numbers in A and also the total number of horizontal index numbers in B, n is the total number of vertical index numbers in B. Further let

$$\rho = \alpha + \beta - p = m - \xi - \eta = n - \lambda - \mu.$$

First suppose that $\rho > 0$. Then when hemipteric matrices of the first type are represented by their hemipteric class-symbols, we can write

$$X = AB$$

$$\begin{pmatrix} a_1a_2 \dots a_\lambda & b_1b_2 \dots b_\rho & c_1c_2 \dots c_\eta & \\ & p_1p_2 \dots p_\rho & q_1q_2 \dots q_\eta & r_1r_2 \dots r_\xi \end{pmatrix} \begin{pmatrix} u_1u_2 \dots u_\mu & v_1v_2 \dots v_\lambda & w_1w_2 \dots w_\rho & \\ & a_1a_2 \dots a_\lambda & b_1b_2 \dots b_\rho & c_1c_2 \dots c_\eta \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} b_1b_2 \dots b_\rho & & \\ p_1p_2 \dots p_\rho & q_1q_2 \dots q_\eta & r_1r_2 \dots r_\xi \end{pmatrix} \begin{pmatrix} u_1u_2 \dots u_\mu & v_1v_2 \dots v_\lambda & w_1w_2 \dots w_\rho \\ & & b_1b_2 \dots b_\rho \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} u_1u_2 \dots u_\mu & v_1v_2 \dots v_\lambda & w_1w_2 \dots w_\rho & & \\ & & p_1p_2 \dots p_\rho & q_1q_2 \dots q_\eta & r_1r_2 \dots r_\xi \end{pmatrix}. \tag{13}$$

In passing from (11) to (12) we have cancelled the first λ zero vertical minors of A with the first λ horizontal minors of B by striking out the index numbers $a_1, a_2, \dots a_\lambda$, and we have cancelled the last η zero horizontal minors of B with the last η vertical minors of A by striking out the index numbers $c_1, c_2, \dots c_\eta$. We pass from (12) to (13) by observing that X must contain $\xi + \eta$ final zero horizontal minors and $\lambda + \mu$ initial zero vertical minors, and must be a hemipteric matrix of the first type whose effective part is the pure hemipteric matrix X' of the first type given by

$$X' = \begin{pmatrix} w_1w_2 \dots w_\rho \\ p_1p_2 \dots p_\rho \end{pmatrix} = \begin{pmatrix} b_1b_2 \dots b_\rho \\ p_1p_2 \dots p_\rho \end{pmatrix} \begin{pmatrix} w_1w_2 \dots w_\rho \\ b_1b_2 \dots b_\rho \end{pmatrix} = A'B',$$

where A' and B' are unbroken parametric diagonal super-minors of A and B, which must be hemipteric matrices of the first type.

From (13) we see that when $\rho > 0$:

(1) The horizontal nullity of X is in general equal to and can never be less than $\xi+\eta$, the sum of the horizontal nullities of A and B.

(2) The vertical nullity of X is in general equal to and can never be less than $\lambda+\mu$, the sum of the vertical nullities of A and B.

(3) The parametric rank of X is in general equal to and can never be greater than ρ; consequently it can never be greater than the parametric rank α of A, and it can never be greater than the parametric rank β of B.

Next suppose that $\rho \ngtr 0$, and let

$$\sigma=-\rho=p-\alpha-\beta=\xi+\eta-m=\lambda+\mu-n>0.$$

Then when hemipteric matrices of the first type are represented by their hemipteric class-symbols, we can write

$$\begin{aligned} X&=AB\\ &=\left(\begin{array}{llll} a_1a_2\dots a_\beta & b_1b_2\dots b_\sigma & c_1c_2\dots c_\alpha & \\ & & p_1p_2\dots p_\alpha & q_1q_2\dots q_\xi \end{array}\right)\left(\begin{array}{llll} u_1u_2\dots u_\mu & v_1v_2\dots v_\beta & & \\ & a_1a_2\dots a_\beta & b_1b_2\dots b_\sigma & c_1c_2\dots c_\alpha \end{array}\right)\\ &=0; \end{aligned}$$

for we can cancel the non-zero horizontal minors of B, which are represented by $a_1, a_2, \dots a_\beta$, with the first β zero vertical minors of A, or we can cancel the non-zero vertical minors of A, which are represented by $c_1, c_2, \dots c_\alpha$, with the last α zero horizontal minors of B. Thus when $\rho \ngtr 0$, X is a zero matrix.

Ex. vii. *Properties of any commutantal product of hemipteric matrices.*

The product matrix X in any commutantal product

$$ABC\dots KL=X$$

of hemipteric matrices has the following properties:

(1) *If X is not a zero matrix, its horizontal nullity is in general equal to and can never be less than the sum of the horizontal nullities of the factor matrices $A, B \dots L$.*

(2) *If X is not a zero matrix, its vertical nullity is in general equal to and can never be less than the sum of the vertical nullities of the factor matrices $A, B, \dots L$.*

(3) *The parametric rank of X cannot be greater than the parametric rank of any one of the factor matrices $A, B, \dots L$.*

It follows by induction from Ex. vi that these results are true for all commutantal products of hemipteric matrices of the first type; and because they must be true for all correlated commutantal products of hemipteric matrices, they must be true generally.

5. Commutantal transformations of a compound matrix by pure hemipteric matrices.

In the following theorem X and Y are compound matrices which both have m horizontal and n vertical index numbers, H is a pure hemipteric matrix having m horizontal and m vertical index numbers, and K is a pure hemipteric matrix having n horizontal and n vertical index numbers. The theorem relates to two different commutantal equations, the first of which is a commutantal transformation. In each equation the types ascribed to X and Y are uniquely determined by the given types of H and K, and therefore the apexes and bases of X and Y are known. Corresponding basical super-minors of X and Y (which by definition are corranged and unbroken) are formed by striking out equal numbers of initial horizontal parametric minors and equal numbers of final vertical parametric minors.

Theorem. Let $HXK = Y$ or $HX = YK$ *be a commutantal equation in which H and K are pure hemipteric matrices, and X and Y are compound matrices; and let X' and Y' be corresponding basical super-minors of X and Y determined by their last μ horizontal parametric minors and their first ν vertical parametric minors. Then by equating the basical super-minors of both sides which correspond to X' and Y' we can deduce from the given equation a commutantal equation*

$$H'X'K' = Y' \quad \text{or} \quad H'X' = Y'K'$$

of the same character and type in which H' is the parametric diagonal super-minor of H determined by its last μ parametric diagonal constituents, and K' is the parametric diagonal super-minor of K determined by its first ν parametric diagonal constituents.

Each of the proposed equations can be derived by class-reversals from a correlated equation of the same character of any one of the 16 possible types. Consequently the theorem will be true generally if it is true generally for an equation of any selected type, and in proving the theorem we may and will suppose that all the matrices are of the first type $\{\pi, \pi\}$. We can then abbreviate the proof by representing each matrix by its class-symbol, the class-symbols of H and K being hemipteric, and utilising the special properties of hemipteric class-symbols of the first type; further the qualifying term 'parametric,' which indicates the order of counting, can be omitted in speaking of the horizontal and vertical minors.

PROOF FOR THE EQUATION $HXK=Y$ (A)

When the type of (A) is $\{\pi, \pi\}\{\pi, \pi\}\{\pi, \pi\}=\{\pi, \pi\}$, (1)

and each matrix is replaced by its class-symbol, let the equation become

$$\begin{pmatrix} u_1u_2 \ldots u_m \\ p_1p_2 \ldots p_m \end{pmatrix}\begin{pmatrix} v_1v_2 \ldots v_n \\ u_1u_2 \ldots u_m \end{pmatrix}\begin{pmatrix} q_1q_2 \ldots q_n \\ v_1v_2 \ldots v_n \end{pmatrix} = \begin{pmatrix} q_1q_2 \ldots q_n \\ p_1p_2 \ldots p_m \end{pmatrix},$$

and let

$$[p_1p_2 \dots p_m] = [p_1p_2 \dots p_r\ P_1P_2 \dots P_\mu],\quad [q_1q_2 \dots q_n] = [q_1q_2 \dots q_\nu\ Q_1Q_2 \dots Q_s],$$

$$[u_1u_2 \dots u_m] = [u_1u_2 \dots u_r\ U_1U_2 \dots U_\mu],\quad [v_1v_2 \dots v_n] = [v_1v_2 \dots v_\nu\ V_1V_2 \dots V_s],$$

where $$r+\mu=m,\quad \nu+s=n,$$

so that, as far as their class-symbols are concerned, we have

$$X' = \begin{pmatrix} v_1\ v_2 \dots v_\nu \\ U_1U_2 \dots U_\mu \end{pmatrix},\quad Y' = \begin{pmatrix} q_1\ q_2 \dots q_\nu \\ P_1P_2 \dots P_\mu \end{pmatrix}.$$

Then from the equation $$HXK=Y \tag{A}$$
or

$$\begin{pmatrix} u_1 \dots u_r\ U_1 \dots U_\mu \\ p_1 \dots p_r\ P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu\ V_1 \dots V_s \\ u_1 \dots u_r\ U_1 \dots U_\mu \end{pmatrix}\begin{pmatrix} q_1 \dots q_\nu\ Q_1 \dots Q_s \\ v_1 \dots v_\nu\ V_1 \dots V_s \end{pmatrix} = \begin{pmatrix} q_1 \dots q_\nu\ Q_1 \dots Q_s \\ p_1 \dots p_r\ P_1 \dots P_\mu \end{pmatrix} \tag{a}$$

we deduce the equation $$H_1XK_1=Y' \tag{A$_1$}$$

or $$\begin{pmatrix} u_1 \dots u_r\ U_1 \dots U_\mu \\ P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu\ V_1 \dots V_s \\ u_1 \dots u_r\ U_1 \dots U_\mu \end{pmatrix}\begin{pmatrix} q_1 \dots q_\nu \\ v_1 \dots v_\nu\ V_1 \dots V_s \end{pmatrix} = \begin{pmatrix} q_1 \dots q_\nu \\ P_1 \dots P_\mu \end{pmatrix} \tag{a$_1$}$$

by striking out the index numbers $p_1, p_2, \dots p_r$ and $Q_1, Q_2, \dots Q_s$ in (a) to represent the striking out of the first r horizontal (active) minors and the last s vertical (active) minors on both sides of (A); and from the equation (A$_1$) we deduce the equation

$$H'X'K'=Y' \tag{A$'$}$$

or $$\begin{pmatrix} U_1U_2 \dots U_\mu \\ P_1P_2 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1\ v_2 \dots v_\nu \\ U_1U_2 \dots U_\mu \end{pmatrix}\begin{pmatrix} q_1q_2 \dots q_\nu \\ v_1v_2 \dots v_\nu \end{pmatrix} = \begin{pmatrix} q_1\ q_2 \dots q_\nu \\ P_1P_2 \dots P_\mu \end{pmatrix} \tag{a$'$}$$

by cancelling the index numbers $u_1, u_2, \dots u_r$ in (a_1) to represent cancellations of the r initial vertical (passive) zero minors of H_1 with the first r horizontal minors of X in (A$_1$), and also cancelling the index numbers $V_1, V_2, \dots V_s$ in (a_1) to represent cancellations of the s final horizontal (passive) zero minors of K_1 with the last s vertical minors of X.

The equations (A), (A$_1$), (A$'$) are all commutantal of the type (1); the matrices H, H_1, H', K, K_1, K' and their class-symbols are hemipteric; the matrices X, X', Y, Y' and their class-symbols are ordinary compound matrices and ordinary class-symbols. In obtaining the equation (A$'$) we have proved the theorem for commutantal equations (A) of the type (1); and it follows by the use of correlations that the theorem is true for commutantal equations (A) of all types. We can adapt the above proof to commutantal equations (A) of types other than (1) by speaking of 'parametric minors' instead of 'minors' to indicate the order of counting, the representations of (A), (A$_1$), (A$'$) by (a), (a_1), (a') being omitted or modified to suit the type.

Ex. i. If we replace the equation (A) of type (1) by a correlated equation of any other type formed by applying class-reversants to H, K, X, Y, then (A$'$) must be replaced by the correlated equation of the same type formed by applying the corresponding class-reversants to H', K', X', Y'. Thus when (A) has the type

$$\{\pi', \pi'\}\{\pi', \pi\}\{\pi, \pi'\} = \{\pi', \pi'\},$$

we replace (a), (a') respectively by

$$\begin{pmatrix} U_\mu \dots U_1\ u_r \dots u_1 \\ P_\mu \dots P_1\ p_r \dots p_1 \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu\ V_1 \dots V_s \\ U_\mu \dots U_1\ u_r \dots u_1 \end{pmatrix}\begin{pmatrix} Q_s \dots Q_1\ q_\nu \dots q_1 \\ P_\mu \dots P_1\ p_r \dots p_1 \end{pmatrix} = \begin{pmatrix} Q_s \dots Q_1\ q_\nu \dots q_1 \\ P_\mu \dots P_1\ p_r \dots p_1 \end{pmatrix},$$

$$\begin{pmatrix} U_\mu \dots U_2U_1 \\ P_\mu \dots P_2P_1 \end{pmatrix}\begin{pmatrix} v_1v_2 \dots v_\nu \\ U_\mu \dots U_2U_1 \end{pmatrix}\begin{pmatrix} q_\nu \dots q_2q_1 \\ v_1v_2 \dots v_\nu \end{pmatrix} = \begin{pmatrix} q_\nu \dots q_2\ q_1 \\ P_\mu \dots P_2P_1 \end{pmatrix}.$$

PROOF FOR THE EQUATION $$HX=YK \tag{B}$$

When the type of (B) is $$\{\pi, \pi\}\{\pi, \pi\}=\{\pi, \pi\}\{\pi, \pi\} \tag{2}$$

and each matrix is replaced by its class-symbol, let the equation become

$$\begin{pmatrix} u_1u_2 \dots u_m \\ p_1p_2 \dots p_m \end{pmatrix}\begin{pmatrix} v_1v_2 \dots v_n \\ u_1u_2 \dots u_m \end{pmatrix} = \begin{pmatrix} q_1q_2 \dots q_n \\ p_1p_2 \dots p_m \end{pmatrix}\begin{pmatrix} v_1v_2 \dots v_n \\ q_1q_2 \dots q_m \end{pmatrix};$$

and again let

$$[p_1p_2 \dots p_m]=[p_1p_2 \dots p_r\ P_1P_2 \dots P_\mu],\quad [q_1q_2 \dots q_n]=[q_1q_2 \dots q_\nu\, Q_1Q_2 \dots Q_s],$$

$$[u_1u_2 \dots u_m]=[u_1u_2 \dots u_r\, U_1U_2 \dots U_\mu],\quad [v_1v_2 \dots v_n]=[v_1v_2 \dots v_\nu\, V_1V_2 \dots V_s].$$

Then from the equation $$HX=YK \tag{B}$$
or

$$\begin{pmatrix} u_1 \dots u_r\, U_1 \dots U_\mu \\ p_1 \dots p_r\, P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu\, V_1 \dots V_s \\ u_1 \dots u_r\, U_1 \dots U_\mu \end{pmatrix} = \begin{pmatrix} q_1 \dots q_\nu\, Q_1 \dots Q_s \\ p_1 \dots p_r\, P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu\, V_1 \dots V_s \\ q_1 \dots q_\nu\, Q_1 \dots Q_s \end{pmatrix} \tag{b}$$

we deduce the equation $$H_1X_1=Y_1K_1 \tag{B$_1$}$$
or

$$\begin{pmatrix} u_1 \dots u_r & U_1 \dots U_\mu \\ & P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu & \\ u_1 \dots u_r & U_1 \dots U_\mu \end{pmatrix} = \begin{pmatrix} q_1 \dots q_\nu & Q_1 \dots Q_s \\ & P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1 \dots v_\nu & \\ q_1 \dots q_\nu & Q_1 \dots Q_s \end{pmatrix} \tag{b$_1$}$$

by striking out the index numbers $p_1, p_2, \dots p_r$ and $V_1, V_2, \dots V_s$ in (b) to represent the striking out of the first r horizontal (active) minors and the last s vertical (active) minors on both sides of (B); and from (B$_1$) we deduce the equation

$$H'X' = Y'K' \tag{B$'$}$$

or

$$\begin{pmatrix} U_1U_2 \dots U_\mu \\ P_1P_2 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1\ v_2 \dots v_\nu \\ U_1U_2 \dots U_\mu \end{pmatrix} = \begin{pmatrix} q_1\, q_2 \dots q_\nu \\ P_1P_2 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_1v_2 \dots v_\nu \\ q_1q_2 \dots q_\nu \end{pmatrix} \tag{b$'$}$$

by cancelling the index numbers $u_1, u_2, \dots u_r$ in (b_1) to represent cancellations of the r initial vertical (passive) minors of H_1 with the first r horizontal minors of X_1 in (B$_1$), and also cancelling the index numbers $Q_1, Q_2 \dots Q_s$ in (b_1) to represent cancellations of the s final horizontal (passive) minors of K_1 with the last s vertical minors of Y_1 in (B$_1$).

In obtaining the equation (B$'$) we have proved the theorem for commutantal equations (B) of the type (2); and it follows by the use of correlations that the theorem is true for commutantal equations (B) of all types.

Ex. ii. If we replace the equation (B) of type (2) by a correlated equation of any other type formed by applying class-reversants to H, K, X, Y, then (B$'$) must be replaced by the correlated equation of the same type formed by applying the corresponding class-reversants to H', K', X', Y'. Thus when (B) has the type

$$\{\pi, \pi'\}\{\pi', \pi'\} = \{\pi, \pi\}\{\pi, \pi'\}$$

we replace (b) and (b') respectively by

$$\begin{pmatrix} U_\mu \dots U_1\, u_r \dots u_1 \\ p_1 \dots p_r\, P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} V_s \dots V_1\, v_\nu \dots v_1 \\ U_\mu \dots U_1\, u_r \dots u_1 \end{pmatrix} = \begin{pmatrix} q_1 \dots q_\nu\, Q_1 \dots Q_s \\ p_1 \dots p_r\, P_1 \dots P_\mu \end{pmatrix}\begin{pmatrix} V_s \dots V_1\, v_\nu \dots v_1 \\ q_1 \dots q_\nu\, Q_1 \dots Q_s \end{pmatrix},$$

$$\begin{pmatrix} U_\mu \dots U_2\, U_1 \\ P_1P_2 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_\nu \dots v_2\ v_1 \\ U_\mu \dots U_2U_1 \end{pmatrix} = \begin{pmatrix} q_1\, q_2 \dots q_\nu \\ P_1P_2 \dots P_\mu \end{pmatrix}\begin{pmatrix} v_\nu \dots v_2v_1 \\ q_1q_2 \dots q_\nu \end{pmatrix}.$$

NOTE.—*Special case in which H and K are undegenerate quadrate hemipteric matrices.*

When H and K are quadrate, we must make the substitutions

$$[u_1\, u_2 \ldots u_m]=[p_1\, p_2 \ldots p_m],\ [v_1\, v_2 \ldots v_n]=[q_1\, q_2 \ldots q_n],$$
$$[U_1 U_2 \ldots U_\mu]=[P_1 P_2 \ldots P_\mu],\ [V_1 V_2 \ldots V_\nu]=[Q_1 Q_2 \ldots Q_\nu]$$

in the proofs of the theorem; and H' and K' are also quadrate. If in addition H and K are undegenerate H' and K' are also undegenerate. Hence we have the following important principle: —

Let H and K in the theorem be undegenerate quadrate hemipteric matrices; and let X' and Y' be correspondingly formed basical super-minors of the compound matrices X and Y. Then X' and Y' have equal ranks; in particular if X' is a zero matrix, then Y' is also a zero matrix, and if Y' is a zero matrix, then X' is also a zero matrix.

6. Commutantal transformations of a pure hemipteric matrix by pure hemipteric matrices.

In the following theorem H, K, X, Y are all pure hemipteric matrices, each having m horizontal and m vertical index numbers. The theorem relates to two different commutantal equations, the first of which is a commutantal transformation. Corresponding unbroken parametric diagonal super-minors of X, Y, H, K (which by definition are corranged), are formed by striking out the horizontal and vertical rows of equal numbers of initial parametric diagonal constituents and of equal numbers of final parametric diagonal constituents.

Theorem. *Let* $HXK=Y$ or $HX=YK$ *be a commutantal equation in which X and Y as well as H and K are pure hemipteric matrices; and let X' and Y' be corresponding unbroken parametric diagonal super-minors of X and Y formed by striking out the horizontal and vertical rows of the first r and the last s parametric diagonal constituents. Then by equating the unbroken parametric diagonal super-minors of both sides which correspond to X' and Y' we can deduce from the given equation a commutantal equation*

$$H'X'K'=Y' \quad \text{or} \quad H'X'=Y'K'$$

of the same character and type in which H' and K' are the unbroken parametric diagonal super-minors of H and K which correspond to X' and Y'.

From the properties of correlated equations we see that the theorem will be true generally if it is true generally when each of the equations has any one of its 16 possible types. Hence in proving the theorem we may and will suppose that all the matrices are of the type $\{\pi, \pi\}$; and we will represent each matrix by its hemipteric class-symbol. We can then utilise the special properties of hemipteric class-symbols of the first type; also we can omit the qualifying term 'parametric' in speaking of the horizontal and vertical minors.

PROOF FOR THE EQUATION $\qquad HXK=Y \qquad$ (A)

When the type of (A) is $\qquad \{\pi, \pi\}\{\pi, \pi\}\{\pi, \pi\}=\{\pi, \pi\}, \qquad$ (1)

and each matrix is replaced by its hemipteric class-symbol, let the equation become

$$\begin{pmatrix} u_1u_2\dots u_m \\ p_1p_2\dots p_m \end{pmatrix}\begin{pmatrix} v_1v_2\dots v_m \\ u_1u_2\dots u_m \end{pmatrix}\begin{pmatrix} q_1q_2\dots q_m \\ v_1v_2\dots v_m \end{pmatrix}=\begin{pmatrix} q_1q_2\dots q_m \\ p_1p_2\dots p_m \end{pmatrix},$$

and let

$$[p_1p_2\dots p_m]=[p_1p_2\dots p_r P_1P_2\dots P_\mu p_1'p_2'\dots p_s'],\ [q_1q_2\dots q_m]=[q_1q_2\dots q_rQ_1Q_2\dots Q_\mu q_1'q_2'\dots q_s'],$$

$$[u_1u_2\dots u_m]=[u_1u_2\dots u_rU_1U_2\dots U_\mu u_1'u_2'\dots u_s'],\ [v_1v_2\dots v_m]=[v_1v_2\dots v_rV_1V_2\dots V_\mu v_1'v_2'\dots v_s'],$$

where $\qquad r+\mu+s=m,$

so that, as far as their class-symbols are concerned, we have

$$X'=\begin{pmatrix} V_1V_2\dots V_\mu \\ U_1U_2\dots U_\mu \end{pmatrix}, \quad Y'=\begin{pmatrix} Q_1Q_2\dots Q_\mu \\ P_1P_2\dots P_\mu \end{pmatrix}.$$

Then from the equation $\qquad HXK=Y \qquad$ (A)

or
$$\begin{pmatrix} u_1\dots u_rU_1\dots U_\mu u_1'\dots u_s' \\ p_1\dots p_rP_1\dots P_\mu p_1'\dots p_s' \end{pmatrix}\begin{pmatrix} v_1\dots v_r V_1\dots V_\mu v_1'\dots v_s' \\ u_1\dots u_rU_1\dots U_\mu u_1'\dots u_s' \end{pmatrix}\begin{pmatrix} q_1\dots q_r Q_1\dots Q_\mu q_1'\dots q_s' \\ v_1\dots v_r V_1\dots V_\mu v_1'\dots v_s' \end{pmatrix}$$
$$=\begin{pmatrix} q_1\dots q_r Q_1\dots Q_\mu q_1'\dots q_s' \\ p_1\dots p_rP_1\dots P_\mu p_1'\dots p_s' \end{pmatrix} \qquad (a)$$

we deduce the equation $\qquad H_1XK_1=Y_1 \qquad$ (A₁)

or
$$\left(\begin{array}{r} u_1\dots u_r U_1\dots U_\mu u_1'\dots u_s' \\ P_1\dots P_\mu p_1'\dots p_s' \end{array}\right)\begin{pmatrix} v_1\dots v_r V_1\dots V_\mu v_1'\dots v_s' \\ u_1\dots u_r U_1\dots U_\mu u_1'\dots u_s' \end{pmatrix}\left(\begin{array}{l} q_1\dots q_r Q_1\dots Q_\mu \\ v_1\dots v_r V_1\dots V_\mu v_1'\dots v_s' \end{array}\right)$$
$$=\left(\begin{array}{l} q_1\dots q_r Q_1\dots Q_\mu \\ \hphantom{q_1\dots q_r} P_1\dots P_\mu p_1'\dots p_s' \end{array}\right); \qquad (a_1)$$

from (A₁) we deduce the equation $\qquad H_2X'K_2=Y_1, \qquad$ (A₂)

$$\left(\begin{array}{l} U_1\dots U_\mu \\ P_1\dots P_\mu p_1'\dots p_s' \end{array}\right)\begin{pmatrix} V_1\dots V_\mu \\ U_1\dots U_\mu \end{pmatrix}\left(\begin{array}{r} q_1\dots q_r Q_1\dots Q_\mu \\ V_1\dots V_\mu \end{array}\right)=\left(\begin{array}{l} q_1\dots q_r Q_1\dots Q_\mu \\ \hphantom{q_1\dots q_r} P_1\dots P_\mu p_1'\dots p_s' \end{array}\right); \qquad (a_2)$$

and finally from (A₂) we deduce the equivalent commutantal equation

$$H'X'K'=Y' \qquad (A')$$

or
$$\begin{pmatrix} U_1U_2\dots U_\mu \\ P_1P_2\dots P_\mu \end{pmatrix}\begin{pmatrix} V_1V_2\dots V_\mu \\ U_1U_2\dots U_\mu \end{pmatrix}\begin{pmatrix} Q_1Q_2\dots Q_\mu \\ V_1V_2\dots V_\mu \end{pmatrix}=\begin{pmatrix} Q_1Q_2\dots Q_\mu \\ P_1P_2\dots P_\mu \end{pmatrix}. \qquad (a')$$

All the matrices and their class-symbols are hemipteric; all the equations are commutantal of the type (1); and (A₁), (A₂), (A′) are equivalent equations. We pass from (A) to (A₁) by striking out the first r horizontal (active) and last s vertical (active) minors on both sides of (A). To pass from (A) to (A₁) we cancel in succession the r initial vertical (passive) zero minors of H₁ with the first r horizontal minors of X by striking out $u_1, u_2, \dots u_r$, and the r initial vertical zero minors thus introduced into X

with the first r horizontal minors of K_1 by striking out $v_1, v_2, \ldots v_r$; and also cancelling in succession the s final horizontal (passive) zero minors of K_1 with the last s vertical minors of X by striking out $v_1', v_2', \ldots v_s'$, and the s final horizontal zero minors thus introduced into X with the last s vertical minors of H_1 by striking out $u_1', u_2', \ldots u_s'$. Finally to pass from (A_2) to (A′) we cancel the s final horizontal (active) zero minors of H_2 and Y_1 by striking out $p_1', p_2', \ldots p_s'$; and also cancel the r initial vertical (active) zero minors of K_2 and Y_1 by striking out $q_1, q_2, \ldots q_r$.

In obtaining the equation (A′) we have proved the theorem for commutantal equations (A) of the type (1); and by the use of correlations it follows that the theorem is true for commutantal equations (A) of all types.

PROOF FOR THE EQUATION $\qquad HX=YK. \qquad$ (B)

When the type of (B) is $\qquad \{\pi, \pi\}\{\pi, \pi\}=\{\pi, \pi\}\{\pi, \pi\}, \qquad$ (2)

and each matrix is replaced by its hemipteric class symbol, let the equation become

$$\begin{pmatrix} u_1u_2 \ldots u_m \\ p_1p_2 \ldots p_m \end{pmatrix}\begin{pmatrix} v_1v_2 \ldots v_m \\ u_1u_2 \ldots u_m \end{pmatrix}=\begin{pmatrix} q_1q_2 \ldots q_m \\ p_1p_2 \ldots p_m \end{pmatrix}\begin{pmatrix} v_1v_2 \ldots v_m \\ q_1q_2 \ldots q_m \end{pmatrix},$$

and again let

$$[p_1p_2 \ldots p_m]=[p_1p_2 \quad p_rP_1P_2 \quad p_1'p_2' \quad . p_s'], \; [q_1q_2 \ldots q_m]=[q_1q_2 \ldots q_rQ_1Q_2 \; .. Q_\mu q_1'q_2' \ldots q_s'],$$

$$[u_1u_2 \ldots u_m]=[u_1u_2 \quad u_rU_1U_2 \ldots U_\mu u_1'u_2' \quad . u_s'], \; [v_1v_2 \ldots v_m]=[v_1v_2 \ldots v_rV_1V_2 \; .. V_\mu v_1'v_2' \quad . v_s'].$$

Then from the equation $\qquad HX=YK \qquad$ (B)

$$\begin{pmatrix} u_1 \ldots u_rU_1 \ldots U_\mu u_1' \ldots u_s' \\ p_1 \ldots p_rP_1 \ldots P_\mu p_1' \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \ldots V_\mu b_1' \ldots v_s' \\ u_1 \ldots u_rU_1 \ldots U_\mu u_1' \ldots u_s' \end{pmatrix}$$

$$=\begin{pmatrix} q_1 \ldots q_rQ_1 \ldots Q_\mu q_1' \ldots q_s' \\ p_1 \ldots p_rP_1 \ldots P_\mu p_1' \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \ldots V_\mu v_1' \ldots v_s' \\ q_1 \ldots q_rQ_1 \ldots Q_\mu q_1' \ldots q_s' \end{pmatrix} \qquad (b)$$

we deduce the equation $\qquad H_1X_1=Y_1K_1 \qquad$ (B_1)

or
$$\begin{pmatrix} u_1 \; .. u_rU_1 \ldots U_\mu u_1' \ldots u_s' \\ \quad P_1 \ldots P_\mu p_1' \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \; .. V_\mu \quad \\ u_1 \ldots u_rU_1 \ldots U_\mu u_1 \ldots u_s' \end{pmatrix}$$

$$=\begin{pmatrix} q_1 \ldots q_rQ_1 \ldots Q_\mu q_1' \ldots q_s' \\ \quad P_1 \ldots P_\mu p_1 \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \ldots V_\mu \quad \\ q_1 \ldots q_rQ_1 \ldots Q_\mu q_1' \ldots q_s' \end{pmatrix}; \qquad (b_1)$$

from (B_1) we deduce the equation $\qquad H_2X_2=Y_2K_2 \qquad$ (B_2)

$$\begin{pmatrix} U_1 \ldots U_\mu \quad \\ P_1 \ldots P_\mu p_1' \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \ldots V_\mu \\ \quad U_1 \ldots U_\mu \end{pmatrix}=\begin{pmatrix} Q_1 \ldots Q_\mu \quad \\ P_1 \ldots P_\mu p_1' \ldots p_s' \end{pmatrix}\begin{pmatrix} v_1 \ldots v_rV_1 \ldots V_\mu \\ \quad Q_1 \ldots Q_\mu \end{pmatrix}; \qquad (b_2)$$

and finally from (B_2) we deduce the equivalent commutantal equation

$$H'X'=Y'K' \qquad (\mathrm{B}')$$

$$\begin{pmatrix} U_1U_2 \ldots U_\mu \\ P_1P_2 \ldots P_\mu \end{pmatrix}\begin{pmatrix} V_1V_2 \ldots V_\mu \\ U_1U_2 \ldots U_\mu \end{pmatrix}=\begin{pmatrix} Q_1Q_2 \ldots Q_\mu \\ P_1P_2 \ldots P_\mu \end{pmatrix}\begin{pmatrix} V_1V_2 \ldots V_\mu \\ Q_1Q_2 \ldots Q_\mu \end{pmatrix} \qquad (b')$$

All the matrices and their class-symbols are hemipteric; all the equations are commutantal of the type (2); and (B_1), (B_2), (B′) are equivalent equations. We pass from (B) to (B_1) by striking out the first r horizontal (active) and the last s vertical (active) minors on both sides of (B). To pass from (B_1) to (B_2) we cancel the r initial vertical (passive) zero minors of H_1 with the first r horizontal minors of X_1 by striking out $u_1, u_2, \ldots u_r$, and the s final horizontal (passive) zero minors of X_1 with the last s vertical minors of H_1 by striking out $u_1', u_2', \ldots u_s'$; and also cancel the r initial vertical (passive) zero minors of Y_1 with the first r horizontal minors of K_1 by striking out $q_1, q_2, \ldots q_r$, and the s final horizontal (passive) zero minors of K_1 with the last s vertical minors of Y_1 by striking out $q_1', q_2' \ldots q_s'$. Finally to pass from (B_2) to (B') we cancel the s final horizontal (active) zero minors of H_2 and Y_2 by striking out $p_1', p_2' \ldots p_s'$; and also cancel the r initial vertical (active) zero minors of X' aud K' by striking out $v_1, v_2, \ldots v_r$.

In obtaining the equation (B') we have proved the theorem for commutantal equations (B) of the type (2); and by the use of correlations it follows that the theorem is true for commutantal equations (B) of all types.

NOTE 1.—*Special case in which H and K are undegenerate quadrate hemipteric matrices.*

When H and K are quadrate, we must make the substitutions

$$[u_1u_2 \ldots u_m]=[p_1p_2 \ldots p_m],\ [U_1U_2 \ldots U_\mu]=[P_1P_2 \ldots P_\mu],\ [u_1'u_2' \ldots u_s']=[p_1'p_2' \ldots p_s'],$$
$$[v_1v_2 \ldots v_m]=[q_1q_2 \ldots q_m],\ [V_1V_2 \ldots V_\mu]=[Q_1Q_2 \ldots Q_\mu],\ [v_1'v_2' \ldots v_s']=[q_1'q_2' \ldots q_s']$$

in the proofs of the theorem, and H' and K' are also quadrate. If in addition H and K are undegenerate, H' and K' are also undegenerate; and from the latter fact we obtain the following important result:—

If H and K in the theorem are undegenerate quadrate hemipteric matrices, then every two corresponding unbroken parametric diagonal super-minors X′ and Y′ of the pure hemipteric matrices X and Y must have equal ranks.

NOTE 2.—*General method of solving the equations* (A) *and* (B).

Let an equation (A) or an equation (B) be given in which some of the constituents of H, K, X, Y are given, and the remaining constituents have to be so determined that the equation is satisfied.

We will suppose that each matrix is represented by the standard skeleton appropriate to its type. When either side of the equation is a product, we will regard the product as replaced by its product matrix represented by the standard skeleton appropriate to its type; but we will regard the constituents of the product matrix as expressed in terms of the constituents of H, K, X, Y.

By equating the corresponding unbroken parametric diagonal constituents of the skeletons of orders $1, 2, \ldots \kappa+1, \ldots m$ on both sides of the given equation we obtain $m-1$ sets of equations

$$(c_1), (c_1), \ldots (c_\kappa), \ldots (c_{m-1}), \qquad \text{(C)}$$

the last set (c_{m-1}) consisting solely of the given equation; and these equations are of course together equivalent to the given equation.

Again by equating the apical constituents on both sides in the equations (c_0), (c_1), ... (c_κ), ... (c_{m-1}) we obtain $m-1$ sets of equations

$$(c_0'),\ (c_1'),\ \ldots\ (c_\kappa')\ \ldots\ c'_{m-1}), \qquad (\mathrm{C}')$$

the equations (c_0') being the same as the equations (c_0). Since the equations (c_κ') are those obtained by equating the corresponding parametric constituents of difference-weight κ on both sides of the given equation, the equations (c_0'), (c_1'), ... (c_κ') are together equivalent to the equations (c_κ); in particular the equations (c_0'), (c_1'), ... (c_{m-1}') are together equivalent to the given equation, i.e. the equation (c_{m-1}).

The equations (c_κ) contain the constituents of H, K, X, Y of difference-weights 0, 1, 2, ... κ only; the same is true of the equations (c_κ'); and when the unknown constituents of difference-weights 0, 1, 2, ... $\kappa-1$ have been so determined that the equations (c_0'), (c_1'), ... $(c_{\kappa-1}')$ are all satisfied, i.e. so that the equations $(c_{\kappa-1})$ are all satisfied, the equations (c_κ) are equivalent to and can be replaced by the equations (c_κ').

The usual method of solving the given equation is to determine in succession the unknown constituents of difference-weight 0 so that the equations (c_0') are all satisfied, the unknown constituents of difference-weight 1 so that the equations (c_1') are all satisfied, ... the unknown constituents of difference-weight κ so that the equations (c_κ') are all satisfied, ... the unknown constituent of difference-weight $m-1$ so that the equation (c_{m-1}') is satisfied. At the $(\kappa+1)$th stage, when determining the unknown constituents of difference-weight κ from the equations (c_κ') we consider that all the constituents of difference-weights 0, 1, 2, ... $\kappa-1$ are known, those which were unknown having the values determined in the preceding stages. It is exactly the same thing if we determine the unknown constituents of difference-weights 0, 1, ... κ, ... $m-1$ in succession so as to satisfy the equations (c_0), (c_1), ... (c_κ), ... (c_{m-1}).

Whenever these processes can be carried out, the general solution of the given equation can be found.

In using these methods an equation (A) of the type (1) is represented by the skeleton equation

$$\begin{vmatrix} H_{11} & H_{12} & \ldots & H_{1m} \\ 0 & H_{22} & \ldots & H_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & H_{mm} \end{vmatrix} \begin{vmatrix} X_{11} & X_{12} & \ldots & X_{1m} \\ 0 & X_{22} & \ldots & X_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & X_{mm} \end{vmatrix} \begin{vmatrix} K_{11} & K_{12} & \ldots & K_{1m} \\ 0 & K_{22} & \ldots & K_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & K_{mm} \end{vmatrix} = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1m} \\ 0 & Y_{22} & \ldots & Y_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & Y_{mm} \end{vmatrix}$$

and the equations (c_κ) (c_κ') are then

$$\begin{bmatrix} H_{ii} & \dots & H_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & H_{jj} \end{bmatrix} \begin{bmatrix} X_{ii} & \dots & X_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & X_{jj} \end{bmatrix} \begin{bmatrix} K_{ii} & \dots & K_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & K_{jj} \end{bmatrix} = \begin{bmatrix} Y_{ii} & \dots & Y_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & Y_{jj} \end{bmatrix}, \qquad (c_\kappa)$$

$$[H_{ii} \dots H_{ij}] \begin{bmatrix} X_{ii} & \dots & X_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & X_{jj} \end{bmatrix} \begin{bmatrix} K_{ij} \\ \vdots \\ K_{jj} \end{bmatrix} = Y_{ij}, \qquad (c_\kappa')$$

$$(j = i + \kappa;\ i = 1, 2, \dots m - \kappa;\ \kappa = 0, 1, 2, \dots m - 1).$$

Again an equation (B) of the type (2) is represented by the skeleton-equation

$$\begin{vmatrix} H_{11} H_{12} & \dots & H_{1m} \\ 0\ H_{22} & \dots & H_{2m} \\ \dots & \dots & \dots \\ 0\ 0 & \dots & H_{mm} \end{vmatrix} \begin{vmatrix} X_{11} X_{12} & \dots & X_{1m} \\ 0\ X_{22} & \dots & X_{2m} \\ \dots & \dots & \dots \\ 0\ 0 & \dots & X_{mm} \end{vmatrix} = \begin{vmatrix} Y_{11} Y_{12} & \dots & Y_{1m} \\ 0\ Y_{22} & \dots & Y_{2m} \\ \dots & \dots & \dots \\ 0\ 0 & \dots & Y_{mm} \end{vmatrix} \begin{vmatrix} K_{11} K_{12} & \dots & K_{1m} \\ 0\ K_{22} & \dots & K_{2m} \\ \dots & \dots & \dots \\ 0\ 0 & \dots & K_{mm} \end{vmatrix}$$

and the equations (c_κ), (c_κ') are then

$$\begin{bmatrix} H_{ii} & \dots & H_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & H_{jj} \end{bmatrix} \begin{bmatrix} X_{ii} & \dots & X_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & X_{jj} \end{bmatrix} = \begin{bmatrix} Y_{ii} & \dots & Y_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & Y_{jj} \end{bmatrix} \begin{bmatrix} K_{ii} & \dots & K_{ij} \\ \dots & \dots & \dots \\ 0 & \dots & K_{jj} \end{bmatrix}, \qquad (c_\kappa)$$

$$[H_{ii} \dots H_{ij}] \begin{bmatrix} X_{ij} \\ \vdots \\ X_{jj} \end{bmatrix} = [Y_{ii} \dots Y_{ij}] \begin{bmatrix} K_{ij} \\ \vdots \\ K_{jj} \end{bmatrix} \qquad (c_\kappa'),$$

$$(j = i + \kappa;\ i = 1, 2, \dots m - \kappa;\ \kappa = 0, 1, 2, \dots m - 1).$$

7. Reduction of an undegenerate quadrate hemipteric matrix by equigradent commutantal transformations.

1. *Reduction of any undegenerate quadrate hemipteric matrix* $A = [a]_m^m$.

Theorem I. *If* $A = [a]_m^m$ *and* $B = [b]_m^m$ *are any two given undegenerate quadrate hemipteric matrices of the same class and the same type, it is always possible to determine an equigradent commutantal transformation.*

$$[h]_m^m\, [a]_m^m\, [k]_m^m = [b]_m^m \quad \text{or} \quad HAK = B, \qquad \text{(A)}$$

converting A into B. Either of the matrices H and K can be chosen arbitrarily, provided that it is an undegenerate quadrate hemipteric matrix of the proper class and type, and the other matrix is then uniquely determinate.

When K is given, we obtain H by postfixing in succession on both sides of (A) the inverses of K and A; when H is given, we obtain K by prefixing in succession on both sides of (A) the inverses of H and A. There are 4 possible types of (A), and the proof is independent of the type.

Ex. i. If A is a given undegenerate quadrate ante-hemipteric matrix, we can choose B in Theorem I to be the unit matrix $[1]_m^m$.

If A is a given undegenerate quadrate counter-hemipteric matrix, we can choose B to be the simple reversant $[j]_m^m$.

Ex. ii. By applications of class-reversants we can deduce from (A) an equigradent commutantal transformation converting A into an undegenerate quadrate hemipteric matrix B' of any assigned type which is correlated with B.

2. ***Reduction of an undegenerate symmetric or skew-symmetric ante-hemipteric matrix $A=[a]_m^m$ by a symmetric equigradent commutantal transformation.***

By Art. 4.7 we can always regard A as a quadrate hemipteric matrix, and it is then necessarily quasi-scalaric. Consequently the following theorem is always applicable.

Theorem II. *If $A=[a]_m^m$ and $B=[b]_m^m$ are any two given symmetric or any two given skew-symmetric quadrate ante-hemipteric matrices of the same class and the same type which are both undegenerate, it is always possible to determine a symmetric equigradent commutantal transformation*

$$\overline{\underline{h}}{}_m^m\,[a]_m^m\,[h]_m^m=[b]_m^m \quad \text{or} \quad H'AH=B \tag{B}$$

converting A into B, the transformation (B) being necessarily quasi-scalaric.

The only two possible types of (B) are

$$\{\pi,\pi\}\,\{\pi,\pi\}\,\{\pi,\pi\}=\{\pi,\pi\},\quad \{\pi',\pi'\}\,\{\pi',\pi'\}\,\{\pi',\pi'\}=\{\pi',\pi'\}; \tag{1}$$

and from Ex. v of Art. 4 we see that A, B and H must all be quasi-scalaric. Since a transformation of either of these types can be reduced to an equivalent correlated symmetric equigradent commutantal transformation of the other type by symmetric applications of class-reversants, we may and will suppose (B) to be of the first of the types (1); and we will further suppose that the equations obtained by replacing A, B and H by their hemipteric class-symbols and their skeletons are respectively

$$\begin{pmatrix}p_1p_2\cdots p_r\\ p_1p_2\cdots p_r\end{pmatrix}\begin{pmatrix}p_1p_2\cdots p_r\\ p_1p_2\cdots p_r\end{pmatrix}\begin{pmatrix}p_1p_2\cdots p_r\\ p_1p_2\cdots p_r\end{pmatrix}=\begin{pmatrix}p_1p_2\cdots p_r\\ p_1p_2\cdots p_r\end{pmatrix}, \tag{B$_1$}$$

$$\left|\begin{matrix} H_{11}' & 0 & \dots & 0 \\ 0 & H_{22}' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & H_{rr}' \end{matrix}\right| \left|\begin{matrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{rr} \end{matrix}\right| \left|\begin{matrix} H_{11} & 0 & \dots & 0 \\ 0 & H_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & H_{rr} \end{matrix}\right| = \left|\begin{matrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_{rr} \end{matrix}\right|, \quad (\mathrm{B}_2)$$

where H_{ii}' is the conjugate of the constituent $H_{ii}=[H_{ii}]^{p_j}_{p_i}$ of H. The equation (B_2) which represents (B) is equivalent to the r equations

$$H_{ii}'A_{ii}H_{ii}=B_{ii}, (i=1, 2, \dots r). \qquad (b_0)$$

The proofs of Theorem II in the two different cases are as follows:—

CASE I. *When A and B are symmetric.*

In this case A_{ii} and B_{ii} are given undegenerate symmetric matrices; $p_1, p_2, \dots p_r$ can be any r non-zero positive integers; and each of the equations (b_0) has the form

$$\overline{[x]}^{p}_{p}\,[a]^{p}_{p}\,[x]^{p}_{p}=[b]^{p}_{p}, \qquad (b_0')$$

where $[a]^p_p$ and $[b]^p_p$ are given undegenerate symmetric matrices. The equation (b_0') can be solved for $[x]^p_p$, (see Theorem IV of § 147 in *Matrices and Determinoids*, Vol. II), and every solution is necessarily an undegenerate square matrix. Thus we can determine the diagonal constituents H_{ii} of H (all of them being necessarily undegenerate) so that all the equations (b_0) are satisfied; and then (B) is a transformation of the required character.

CASE II. *When A and B are skew-symmetric.*

In this case A_{ii} and B_{ii} are given undegenerate skew-symmetric matrices; $p_1, p_2, \dots p_r$ must be *even* positive integers; and each of the equations (b_0) has the form (b_0'), where p is an even positive integer, and $[a]^p_p$ and $[b]^p_p$ are given undegenerate skew-symmetric matrices. The equation (b_0') can be solved for $[x]^p_p$, (see Theorem IV of § 150 in *Matrices and Determinoids*, Vol. II), and every solution is necessarily an undegenerate square matrix. Thus we can determine the diagonal constituents H_{ii} of H (all of them being necessarily undegenerate) so that all the equations (b_0) are satisfied; and then (B) is a transformation of the required character.

Ex. iii. If A is a given undegenerate *symmetric* ante-hemipteric matrix, we can choose B in Theorem II to be the unit matrix $[1]^m_m$.

If A is a given undgenerate *skew-symmetric* ante-hemipteric matrix, so that m is even, we can choose B to be the matrix

$$\begin{bmatrix} \omega, 0, \ldots 0 \\ 0, \omega, \ldots 0 \\ \cdots\cdots \\ 0, 0, \ldots \omega \end{bmatrix}^{2, 2, \ldots 2}_{2, 2, \ldots 2} \quad \text{in which } [\omega]^2_2 = \begin{bmatrix} 0, 1 \\ -1, 0 \end{bmatrix}.$$

Ex. iv. If A is a given undegenerate symmetric (or skew-symmetric) ante-hemipteric matrix of the class $\begin{pmatrix} p_1, p_2, \ldots p_r \\ p_1, p_2, \ldots p_r \end{pmatrix}$, then by symmetric applications of class-reversants to the two sides of (B) we can obtain a symmetric equigradent commutantal transformation, necessarily quasi-scalaric, converting A into any undegenerate symmetric (or skew-symmetric) ante-hemipteric matrix B' of the class $\begin{pmatrix} p_r, \ldots p_2, p_1 \\ p_r, \ldots p_2, p_1 \end{pmatrix}$ of the type conjugate to that of A. Of course $p_1, p_2, \ldots p_r$ must all be even, and B' must be quasi-scalaric.

3. *Reduction of an undegenerate symmetric or skew-symmetric counter-hemipteric matrix $A = [a]^m_m$ by a symmetric equigradent commutantal transformation.*

By Art. 4. 7 we can always regard A as a quadrate hemipteric matrix of a symmetric class. Consequently the following theorem is always applicable.

Theorem III. *If $A = [a]^m_m$ and $B = [b]^m_m$ are any two given symmetric or any two given skew-symmetric counter-hemipteric matrices of the same symmetric class and of the same type which are both undegenerate, it is always possible to determine a symmetric equigradent commutantal transformation*

$$\overline{\underline{h}}{}^{m}_{m}[a]^m_m[h]^m_m = [b]^m_m \quad \text{or} \quad H'AH = B \tag{C}$$

converting A into B. The constituents of H lying on one side of the median line can be chosen arbitrarily subject to the condition that those of them which are parametric diagonal constituents are undegenerate.

We shall now follow the general method described in Note 2 of Art. 6, which could also have been followed in proving Theorems I and II.

The only two possible types of (C) are

$$\{\pi', \pi\}\{\pi', \pi\}\{\pi, \pi\} = \{\pi', \pi\}, \quad \{\pi, \pi\}\{\pi, \pi'\}\{\pi', \pi'\} = \{\pi, \pi'\} \tag{2}$$

Since a transformation (C) of either one of these types can be converted into an equivalent correlated symmetric equigradent commutantal transformation of the other type by symmetric applications of class-reversants, we may and will suppose (C) to have the first of the types (2); and we will further suppose that the equations derived from (C) by replacing A, B and H by their hemipteric class-symbols and by their skeletons are respectively

$$\begin{pmatrix} p_1 p_2 \cdots p_r \\ p_1 p_2 \cdots p_r \end{pmatrix}\begin{pmatrix} p_1 p_2 \cdots p_r \\ p_1 p_2 \cdots p_r \end{pmatrix}\begin{pmatrix} p_1 p_2 \cdots p_r \\ p_1 p_2 \cdots p_r \end{pmatrix}=\begin{pmatrix} p_1 p_2 \cdots p_r \\ p_1 p_2 \cdots p_r \end{pmatrix}, \qquad (\mathrm{C}_1)$$

$$\left|\begin{matrix} H_{11}' & 0 & \ldots & 0 \\ H_{12}' & H_{22}' & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ H_{1r}' & H_{2r}' & \ldots & H_{rr}' \end{matrix}\right| \left|\begin{matrix} 0 & 0 & \ldots & A_{rr} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & A_{22} & \ldots & A_{2r} \\ A_{11} & A_{12} & \ldots & A_{1r} \end{matrix}\right| \begin{bmatrix} H_{11} & H_{12} & \ldots & H_{1r} \\ 0 & H_{22} & \ldots & H_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & H_{rr} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & B_{rr} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & B_{22} & \ldots & B_{2r} \\ B_{11} & B_{12} & \ldots & B_{1r} \end{bmatrix} \qquad (\mathrm{C}_2)$$

where $$p_{r+1-i}=p_i, \qquad (3$$

and $$A_{ij}=[A_{ij}]_m^{p_j},\quad B_{ij}=[B_{ij}]_m^{p_j},\quad H_{ij}=[H_{ij}]_m^{p_j},\quad H_{ij}'=\overline{H_{ij}}\,{}^{p_i}_{p_j}. \qquad (4$$

We will define integers u and v by the equations

$$i+u=j+v=r+1,\ (i=1, 2, \ldots r;\ j=1, 2, \ldots r); \qquad (5)$$

and we will regard (C) as represented by the skeleton-equation (C_2). We can replace (C) or (C_2) by the r sets of equations (c_0), (c_1), … (c_κ), … (c_{r-1}) which are obtained by equating corresponding unbroken parametric diagonal super-minors on both sides, and which are given by

$$H_{uu}'A_{ii}H_{ii}=B_{ii},\ (i=1, 2, \ldots r); \qquad (\mathrm{c}_0)$$

$$\left|\begin{matrix} H_{vv}' & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ H'_{v,u-1} & \ldots & H'_{u-1,u-1} & 0 \\ H_{vu}' & \ldots & H'_{u-1,u} & H_{uu}' \end{matrix}\right| \left|\begin{matrix} 0 & 0 & \ldots & A_{jj} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & A_{i+1,i+1} & \ldots & A_{i+1,j} \\ A_{ii} & A_{i,i+1} & \ldots & A_{ij} \end{matrix}\right| \left|\begin{matrix} H_{ii} & H_{i,i+1} & \ldots & H_{ij} \\ 0 & H_{i+1,i+1} & \ldots & H_{i+1,j} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & H_{jj} \end{matrix}\right|$$

$$= \left|\begin{matrix} 0 & 0 & \ldots & B_{jj} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & B_{i+1,i+1} & \ldots & B_{i+1,j} \\ B_{ii} & B_{i,i+1} & \ldots & B_{ij} \end{matrix}\right|, \qquad (c_\kappa)$$

$$(j=i+\kappa;\ i=1, 2, \ldots r-\kappa;\ \kappa=1, 2, \ldots r-1);$$

the last equation (c_{r-1}) being (C_2). We can also replace the equations (c_0), (c_1), … (c_{r-1}) by the r sets of equations (c_0'), (c_1'), … (c_{r-1}') derived from them by equating the apical constituents on both sides, the equations (c_0') being the same as (c_0), and the equations (c_κ') being

$$[H_{ru}' \ldots H_{uu}'] \begin{bmatrix} 0 & \ldots & A_{jj} \\ \ldots & \ldots & \ldots \\ A_{ii} & \ldots & A_{ij} \end{bmatrix} \begin{bmatrix} H_{ij} \\ \vdots \\ H_{jj} \end{bmatrix} = B_{ij}. \qquad (c_\kappa')$$

We will denote the conjugates of A_{ij}, B_{ij}, A_{ru}, B_{ru} by A_{ij}', B_{ij}', A_{ru}', B_{ru}'. The proofs of Theorem III in the two different cases are as follows :—

CASE I. *When A and B are both symmetric.*

In this case each of the matrices A and B is symmetric with respect to its median line, the constituents of the median line itself being symmetric, i.e. the constituents of A and B are such that

$$A_{ij} = A_{ru}', \quad B_{ij} = B_{ru}',$$

$$A_{ij} \text{ and } B_{ij} \text{ are symmetric when } i + j = r + 1.$$

The positive integers $p_1, p_2, \ldots p_r$ can have any values consistent with (3).

First let the r successive equations (c_0) be called $E_1, E_2, \ldots E_r$; and let

$$r = 2t \text{ or } r = 2t + 1 \text{ according as } r \text{ as even or odd.}$$

Then the last t of the equations (c_0) can be omitted as being redundant; for if $i \ngtr t$, the two equations E_i and E_{r+1-i} are mutually conjugate and therefore mutually equivalent, and the second of them is superfluous. Accordingly when r is even, the r equations (c_0) are equivalent to the t equations $E_1, E_2, \ldots E_t$, the equation E_i being

$$H_{uu}'A_{ii}H_{ii} = B_{ii}, \quad (i = 1, 2, \ldots t;\ u > i), \qquad (E_i)$$

and having the form

$$\overline{\underline{y}}{}^{p}_{p}\ [a]^{p}_{p}\ [x]^{p}_{p} = [b]^{p}_{p}, \qquad (E_i')$$

where $[x]^{p}_{p} \doteq H_{ii}$ and $[y]^{p}_{p} = H_{uu}$ are diagonal constituents of H equidistant from the first and last diagonal constituents, and where $[a]^{p}_{p} = A_{ii}$ and $[b]^{p}_{p} = B_{ii}$ are given undegenerate square matrices; and when r is odd, the r equations (c_0) are equivalent to the same t equations $E_1, E_2 \ldots E_t$ together with the one additional middle equation

$$H'_{t+1,t+1} A_{t+1,t+1} H_{t+1,t+1} = B_{t+1,t+1} \qquad (E_{t+1})$$

having the form

$$\overline{\underline{x}}{}^{p}_{p}\ [a]^{p}_{p}\ [x]^{p}_{p} = [b]^{p}_{p}, \qquad (E'_{t+1})$$

where $[x]^{p}_{p} = H_{t+1,t+1}$ is the diagonal constituent of H lying on the median line, and where $[a]^{p}_{p} = A_{t+1,t+1}$ and $[b]^{p}_{p} = B_{t+1,t+1}$ are given undegenerate symmetric matrices.

The equation E_i, $(i = 1, 2, \dots t)$, represented by (E_i') can be solved for H_{ii} and H_{uu}; all solutions are necessarily undegenerate; either of the two matrices H_{ii} and H_{uu} can be chosen arbitrarily, provided that it is undegenerate; and the other is then uniquely determinate. The equations $E_1, E_2, \dots E_t$ are all satisfied when and only when the first t and last t diagonal constituents of H are determined in this way. When r is odd, the remaining equation E_{t+1} can be solved for the remaining middle diagonal constituent $H_{t+1,t+1}$, and every solution is necessarily undegenerate. Thus we can always determine the r constituents of H of difference-weight 0 so that the r equations (c_0) are all satisfied; all those constituents are then necessarily undegenerate square matrices; the middle one (when r is odd) must be so chosen as to satisfy the equation (E_{t+1}'); those of them which lie on one side of the median line of H can be chosen arbitrarily, provided that they are undegenerate; and those of them which lie on the other side of the median line are then uniquely determinate.

We can now complete the proof of the theorem by induction. Supposing κ to have any one of the values $1, 2, \dots r-1$, we will make the hypothesis that the parametric constituents of H of difference-weights $0, 1, \dots \kappa-1$ have been so determined that the equations $(c_0), (c_1), \dots (c_{\kappa-1})$ or the equivalent equations $(c_0'), (c_1'), \dots (c_{\kappa-1}')$ are all satisfied; and we will show that the parametric constituents of H of difference-weight κ can then be so determined that the equations (c_κ) or the equivalent equations (c_κ') are all satisfied, even when those of them which lie on one side of the median line of H are chosen arbitrarily. This will establish the theorem.

Let the $r-\kappa$ successive equations (c_κ) be called $E_1, E_2, \dots E_{r-\kappa}$, and the $r-\kappa$ successive equations (c_κ') be called $e_1, e_2, \dots e_{r-\kappa}$, the equation e_i being equivalent to the equation E_i in consequence of the hypothesis; and let

$$r-\kappa = 2\tau \text{ or } r-\kappa = 2\tau + 1 \text{ according as } r-\kappa \text{ is even or odd.}$$

Then the last τ of the equations (c_κ) or (c_κ') can be omitted as being redundant; for if $i \ngtr \tau$, the equations E_i and $E_{r-\kappa+1-i}$ are mutually conjugate and therefore mutually equivalent, and the second of them is superfluous. Accordingly when $r-\kappa$ is even, the $r-\kappa$ equations (c_κ) or (c_κ') are equivalent to the τ equations $e_1, e_2, \dots e_\tau$, the equation e_i being

$$H_{vu}' \,.\, A_{jj} H_{jj} + H_{uu}' A_{uu} \,.\, H_{ij} = \beta_{ij}, \qquad (e_i)$$
$$(i = 1, 2, \dots \tau;\ j - i = u - v = \kappa;\ v > i,\ u > j;\ \beta_{ij} = \beta_{vu}'),$$

and having the form

$$\overline{\underline{y}}{}^{q}_{p}\ [b]^q_q+\overline{\underline{a}}{}^{p}_{p}\ [x]^q_p=[c]^q_p, \qquad (e_i')$$

where $[a]^p_p=A_{uu}H_{uu}$ and $[b]^q_q=A_{jj}H_{jj}$ are given undegenerate square matrices, and where $[c]^q_p=\beta_{ij}$ is a given matrix because it only involves constituents of H having difference-weight less than κ; and when $r-\kappa$ is odd, the $r-\kappa$ equations (c_κ) or (c_κ') are equivalent to the same τ equations $e_1, e_2, \ldots e_\tau$ together with the one additional middle equation $e_{\tau+1}$ which is derived from (e_i) by putting

$$i=v=\tau+1,\ j=u=\tau+1+\kappa=r-\tau,$$

and is the equation

$$H'_{\tau+1,\,r-\tau}\,.\,A_{r-\tau,\,r-\tau}H_{r-\tau,\,r-\tau}+H'_{r-\tau,\,r-\tau}\,A'_{r-\tau,\,r-\tau}\,.\,H_{\tau+1,\,r-\tau}=\beta_{\tau+1,\,r-\tau} \qquad (e_{\tau+1})$$

having the form

$$\overline{\underline{x}}{}^{p}_{p}\ [a]^p_p+\overline{\underline{a}}{}^{p}_{p}\ [x]^p_p=[c]^p_p, \qquad (e'_{\tau+1})$$

where $[a]^p_p=A_{r-\tau,\,r-\tau}\,H_{r-\tau,\,r-\tau}$ is a given undegenerate square matrix, and $[c]^p_p=\beta_{\tau+1,\,r-\tau}$ is a given symmetric matrix.

The equation e_i, $(i=1, 2, \ldots \tau)$, represented by (e_i') can be solved for H_{ij} and H_{vu}, which are two constituents of the $(\kappa+1)$th parametric diagonal line of H equidistant from the first and last constituents of that line, and either one of those two constituents can be chosen arbitrarily, the other one being then uniquely determinate. The equations $e_1, e_2, \ldots e_\tau$ are all satisfied when and only when the first τ and last τ constituents of the $(\kappa+1)$th parametric diagonal line of H are determined in this way. When $r-\kappa$ is odd, the remaining equation $e_{\tau+1}$ or $e_{\tau+1}'$ can be solved for the remaining middle constituent $H_{\tau+1,\,r-\tau}$ of the $(\kappa+1)$th parametric diagonal line of H by putting

$$[c]^p_p=\overline{\underline{d}}{}^{p}_{p}+[d]^p_p,\ \overline{\underline{a}}{}^{p}_{p}\ [x]^p_p=[d]^p_p.$$

Thus (on the hypothesis which has been made) we can determine the $r-\kappa$ parametric constituents of H of difference-weight κ so that the $r-\kappa$ equations (c_κ') or (c_κ) are satisfied; and those of the constituents which lie on one side of the median line can be chosen arbitrarily.

It follows by induction that Theorem III is true in Case I.

CASE II. *When A and B are both skew-symmetric.*

In this case each of the matrices A and B is skew-symmetric with respect to its median line, the constituents of the median line itself being skew-symmetric, i.e. the constituents of A and B are such that

$$A_{ij} = -A_{vu}', \quad B_{ij} = -B_{vu}',$$

A_{ij} and B_{ij} are skew-symmetric when $i+j=r+1$.

When r is even, the positive integers $p_1, p_2, \dots p_r$ can have any values consistent with (3); but when $r=2t+1$ is odd, the middle integer p_{t+1} must be even, as otherwise A and B could not be undegenerate.

First let the r successive equations (c_0) be called $E_1, E_2, \dots E_r$, and let

$r=2t$ or $r=t+1$ according as r is even or odd.

Then the last t of the equations (c_0) can be omitted as being redundant; for if $i \not> t$, the two equations E_i and E_{r+1-i} are mutually skew-conjugate and therefore mutually equivalent, and the second of them is superfluous. Accordingly when r is even the r equations (c_0) are equivalent to the t equations $E_1, E_2, \dots E_t$, the equation E_i being

$$H_{uu}' A_{ii} H_{ii} = B_{ii}, \quad (i=1, 2, \dots t;\ u>i), \qquad (E_i)$$

and having the same form

$$\overline{\underline{y}}_p^{\,p} [a]_p^p [x]_p^p = [b]_p^p \qquad (E_i')$$

and the same character as in Case I; and when r is odd, the r equations (c_0) are equivalent to the same equations $E_1, E_2 \dots E_t$, together with the one additional middle equation

$$H'_{t+1,t+1} A_{t+1,t+1} H_{t+1,t+1} = B_{t+1,t+1} \qquad (E_{t+1})$$

having the same form

$$\overline{\underline{x}}_p^{\,p} [a]_p^p [x]_p^p = [b]_p^p, \qquad (E_{t+1}')$$

as in Case I, but where now p is an even integer, and $[a]_p^p$ and $[b]_p^p$ are given undegenerate skew-symmetric matrices.

The equations $E_1, E_2, \dots E_t$ can be solved for the first t and last t diagonal constituents of H exactly as in Case I, all solutions being necessarily undegenerate. When r is odd, the remaining equation E_{t+1} can be solved for the remaining middle diagonal constituent of H, and every solution is necessarily undegenerate. Thus we can always determine the r constituents of H of difference-weight 0, so that the r equations (c_0) are all satisfied; all those constituents are then necessarily undegenerate; the middle one (when r is odd) must satisfy the equation E_{t+1}'; those of them which lie on one side of the median line of H can be chosen arbitrarily, provided that they are undegenerate; and those of them which lie on the other side of the median line are then uniquely determinate.

We now make the same hypothesis as in Case I, and consider the equivalent equations (c_κ) and (c_κ'), denoting them as in Case I by E_1, $E_2, \ldots E_{r-\kappa}$ and $e_1, e_2, \ldots e_{r-\kappa}$, and putting

$$r-\kappa=2\tau \text{ or } r-\kappa=2\tau+1 \quad \text{according as } r-\kappa \text{ is even or odd.}$$

The last τ of the equations (c_κ) or (c_κ') can be omitted as being redundant; for if $i \not> \tau$, the equations E_i and $E_{r-\kappa+1-i}$ are mutually skew-conjugate and therefore mutually equivalent, and the second of them is superfluous. Accordingly when $r-\kappa$ is even, the $r-\kappa$ equations (c_κ) or (c_κ') are equivalent to the τ equations $e_1, e_2, \ldots e_\tau$, the equation e_i being

$$H_{vu}'.\,A_{jj}\,H_{jj}-H_{uu}'A_{uu}'.\,H_{ij}=\beta_{ij}, \tag{e_i}$$

$$(i=1, 2, \ldots \tau;\ j-i=u-v=\kappa;\ v>i,\ u>j,\ \beta_{ij}=-\beta_{vu}'),$$

and having the form

$$\overline{\underline{y}}^{\,q}\,[b]_q^q-\overline{\underline{a}}^{\,p}\,[x]_p^q=[c]_p^q, \tag{e_i'}$$

where $[a]_p^p$ and $[b]_q^q$ are given undegenerate square matrices, and $[c]_p^q$ is a given matrix; and when $r-\kappa$ is odd, the $r-\kappa$ equations (c_κ) or (c_κ') are equivalent to the same τ equations $e_1, e_2, \ldots e_\tau$ together with the one additional middle equation $e_{\tau+1}$ or

$$H'_{\tau+1,\,r-\tau}\,.\,A_{r-\tau,\,r-\tau}\,H_{r-\tau,\,r-\tau}-H'_{r-\tau,\,r-\tau}\,A'_{r-\tau,\,r-\tau}\,.\,H_{\tau+1,\,r-\tau}=\beta_{\tau+1,\,r-\tau} \tag{$e_{\tau+1}$}$$

having the form

$$\overline{\underline{x}}^{\,p}_{\,p}\,[a]_p^p-\overline{\underline{a}}^{\,p}_{\,p}\,[x]_p^p=[c]_p^p, \tag{$e_{\tau+1}'$}$$

where $[a]_p^p$ is a given undegenerate square matrix, and $[c]_p^p$ is a given skew-symmetric matrix.

We can always determine the first τ and last τ of the $r-\kappa$ constituents of the $(\kappa+1)$th parametric diagonal line of H so that the equations

$e_1, e_2, \ldots e_\tau$ are all satisfied exactly as in Case I; and when $r-\kappa$ is odd, the remaining equation $e_{\tau+1}$ or $e_{\tau+1}'$ can be solved for the remaining middle constituent of that line by putting

$$[c]^p_p = \overline{\underline{d}}\,^p_p - [d]^p_p, \quad \overline{\underline{a}}\,^p_p\,[x]^p_p = [d]^p_p.$$

Thus (on the hypothesis which has been made) we can always determine the $r-\kappa$ parametric constituents of H of difference-weight κ so that the $r-\kappa$ equations (c_κ') or (c_κ) are all satisfied; and those of the constituents which lie on one side of the median line can be chosen arbitrarily.

It follows by induction that Theorem III is true in Case II.

Ex. v. If A is a given undegenerate symmetric counter-hemipteric matrix, we can choose B in Theorem III to be the simple reversant $[j]^m_m$.

If A is a given undegenerate skew-symmetric counter-hemipteric matrix, and if $m=2\mu$, we can choose B to be the matrix

$$\begin{bmatrix} 0, & j \\ -j, & 0 \end{bmatrix}^{\mu,\ \mu}_{\mu,\ \mu} \text{ in which } [j]^\mu_\mu \text{ is the simple reversant of order } \mu.$$

Ex. vi. By symmetric applications of class-reversants to the two sides of (C) we can obtain a symmetric equigradent commutantal transformation converting the undegenerate symmetric (or skew-symmetric) counter-hemipteric matrix A into any undegenerate symmetric (or skew-symmetric) counter-hemipteric matrix B' of the same symmetric class as A and of the type inverse to that of A.

CEPHALEUROS VIRESCENS, KUNZE, AN ALGA PARASITIC ON THE MANGO TREE AND OTHER FRUIT TREES OF BENGAL.

(With three Plates.)

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Entophytic or parasitic alga, as is well known, are of comparatively rare occurrence. The alga *Cephaleuros virescens*, Kunze, attracted the attention of the tea planters in Assam as far back as 1880. At that time it was commonly termed "White Blight," because the leaves and shoots attacked by the alga became variegated, and in some cases quite white. At this time an interesting account of the alga was published by D. D. Cunningham under the name of *Mycoidea parasitica* in the "Transactions of the Linnean Society" (Ser. 2, Vol. I, 1880, pp. 301–316). A little later H. Marshall Ward published his work on "a Tropical Epiphyllous Lichen" in the Transactions of the Linnean Society (Vol. II, Ser. 2, Botany, 1883, pp. 87–115), where he criticises Cunningham's work. Cunningham gave two other accounts of this alga in two subsequent papers published in the Scientific Memoirs of the Medical Officers in India (part III, 1887 and part X, 1897). It is in the later publication of 1897 that he observed the parasite as a "Bark Blight."

The algal nature of *Cephaleuros* was first worked out in 1824 by Agardh in his "Systema algarum," and in 1891 G. Karsten worked out the whole group of *Chroolepideae*, to which group the genus *Cephaleuros* belongs, and published his work in the Annales du Jardin Botanique de Buitenzorg (Vol. X, 1891, pp. 1–65) under the title "Untersuchungen über die Familie der Chroolepideen." He describes the group of *Chroolepideae* thus:—

"*Chroolepideae.* An der luft lebende, durch den besitz von hämatochrom ausgezeichnete algen, welche aus zellfäden oder zellflächen bestehen und in einzelnen von den vegetativen mehr oder minder abweichenden, zellen eine grosse zahl, mit stets 2 cilien versehner, schwärmer entwickeln."

He then divides this family into 4 genera :—(1) *Trentepohlia*, Mart.; (2) *Chroolepus*, non Agardh; (3) *Phycopeltis*, Millardet; (4) *Cephaleuros*, Kunze.

As we are only concerned here with the genus *Cephaleuros*, Kunze, we may as well quote Karsten's description of the same, which is still accepted as authoritative. Karsten says of the genus *Cephaleuros*:

"Die hauptunterschiede dieser Cephaleuros-gruppe gegen die früher behandelten formen sind:

Mehrschichtigkeit des thallus, bedingt durch unregelmässige aussackungen, die mit Ward als "rhizoiden" bezeichnet sein mögen, schmiegen sich auf das engste ans substrat an und heften den thallus darauf fest.

Die beharung ("barren hairs" Ward) bildet eine weitere differenz und die grossen hakensporangien, deren basalzelle, wie bei *Trentepohlia bisporangiata*, eine grosse zahl von halszellen mit je einem sporangiumköpfchen trägt ("fertile hairs," Ward) sind das zunächst in die augen fallende unterschiedungsmerkmal der gattung Phycopeltis gegenüber."

Cephaleuros virescens, Kunze, as a tea-blight was first investigated by Dr. (now Sir George) Watt in 1898 and later by Watt and Mann in 1903 and still later by Mann and Hutchinson in 1904. In 1907 Mann and Hutchinson published a very interesting account of the algal parasite, as attacking the tea plants, in the "Memoirs of the Department of Agriculture in India" (Bot. Ser., April 1907). These authors state: "Some confusion has arisen as to the proper botanical name to use for this alga. *Cephaleuros virescens* was used by Kunze in 1829 for an alga he described from Guinea, but his description is hardly sufficiently detailed to be quite certain that we are dealing with the same alga, and his type-specimens have disappeared. On this account Karsten in 1891 gave it the new name *Cephaleuros mycoidea* and published an exact description. We cannot think that the reasons given by Karsten are sufficient to justify the abandonment of the older name, and accordingly retain it here. In doing so we have the support of Professor Delacroix of Paris."

We also, in the present article, retain the old name *Cephaleuros virescens*, Kunze. In this connection we acknowledge our thanks to Dr. E. J. Butler of Pusa, who kindly allowed his assistant Babu Rohini Ranjan Sen to confirm the specific determination of the alga, specimens of which were sent to him.

Hitherto the alga has been studied by the more recent investigators, as attacking the tea plants; though it was also noticed on the leaves of mango trees, the only other economic plant which is a prey to the infection of this parasite. Dr. Butler says that the number of host plants of this parasite probably exceeds 100; and he mentions in the "Proceedings of the Board of Agriculture for India" (1906, p. 124), the appearance of this alga on mango twigs as causing serious damage to the mango crops in the Malda District, Bengal.

The alga, having been found to attack the tea plants and mango trees, led us to search for other fruit trees that might be attacked by the same parasite, and it was found by us to attack the following fruit trees besides the mango tree:—

(1) *Artocarpus integrifolia*, Linn.
(2) *Eugenia malaccensis*, Linn.
(3) *Dillenia indica*, Linn.

We also found the parasite on several other plants numbering about thirty including some garden plants such as *Mangolia grandiflora*. It is the attack of the alga on *Mangifera indica*, Linn., that attracted our chief attention, and the following description is chiefly confined to the forms found on that tree. It should be noted here that the alga exists in two forms, (1) as a leaf blight, (2) as a stem blight; and it is only when it occurs as a stem blight that it causes any serious damage to its host.

Cephaleuros, as is characteristic of the family of *Chroolepideae*, is generally epiphytic on leaves and twigs, but not infrequently it is found to be entophytic. In the mango trees, the leaves were found completely pierced by the algal cells, and fertile fructifications appear on both the upper and lower surfaces of the leaves.

The alga, as occurring on mango leaves, was first observed by us in the middle of March 1919. From that time onwards the leaves of several other plants were also found to be attacked. Later, during the rainy season, the alga was found attacking the twigs of the mango trees, though these attacks were few in number as compared to those on the leaves. The patches on the leaves of the mango trees were found to be of lichenoid growth and from one of these lichenlike patches thus found a few sterile and fertile hairs of the alga were found growing out into the air. It suggested itself at once that it might be the *helotic* alga which grew out in suitable conditions and that view has been confirmed by an examination of infested leaves of different host plants. On examination of a good number of mango leaf sections where the lichen was not so old, it was found that a fungus carries the algal cells which possess the haematochrom. The fungus was fructifying and it produced spores inside the pycnidia. But as the lichen grew old and rains set in, the alga grew out into the air and reproduced. Here the condition was not *helotic*, but rather the opposite—the alga predominating, piercing the leaf and producing sporangia on both surfaces of the leaf. This phenomenon was also observed on the infested leaves of *Croton* and *Magnolia*, which plants are among its hosts. From the above observations, it is concluded that the alga has assumed the lichen-forming habit, and it not only grows independently on leaves and twigs, but also for some portion of its life it serves the fungus as a host, forming a lichen.

The alga occurs as orange-yellow to almost red circular patches, generally on the upper surface of the leaves. In *Dillenia*, they were purely yellow in colour, whilst in *Podocarpus* they were yellowish-red. The size of the patches varies greatly. In *Dillenia*, the spots in some cases were more than a centimetre in diameter and on mango leaves they were considerably smaller. These patches, when examined, are found to contain masses of delicate erect hyphae, some of which are sterile, while others are fertile, each bearing a sporangium. The thallus is seen to be composed of more than one row of cells, but its thickness is not uniform, being generally greatest at the centre. The algal thallus is generally superficial, but it is found to penetrate the tissue of the host leaf immediately in touch with the alga, the tissue itself becoming discoloured and when dead turning brown. The pallisade cells of the leaf-tissue are found to be divided by transverse septa, and formation of corky tissues takes place.

Certain algal patches on mango leaves were kept under constant observation, and not a single case of penetration could be found; but with the advent of the rains the alga grew luxuriantly, and the affected leaves were mostly found to be penetrated by it, the fructifications being produced on the lower surface of the leaves. Sometimes whitish patches are observable, which are in fact algal patches devoid of all colouring matter and filled up with air. The orange colour of the alga is invariably due to the presence of the pigment haematochrom.

As has already been mentioned, during the rainy season only the twigs of the mango trees were found to be attacked by the alga. In fact, the twigs of other fruit trees mentioned have been searched in vain for any traces of infection. It was also noticed that no algal patch was found on the green portion of the twigs bearing the mango leaves, and in these cases they were found only on that portion of the stem which lay just below the green portion bearing the leaves. In such cases the leaves on these infested twigs become yellow, and sooner or later the twigs themselves die, whilst those neighbouring twigs which are not infested by the alga bear quite healthy leaves and are themselves in a healthy state. It is, therefore, evident that it is the stem attack which causes serious damage to the plants. The leaf attack itself is harmful for the very simple reason that the infection of the stem, in all probability, is due to the leaf alga.

Reproduction:—The contents of the cells of the sporangium are minute coloured granules, the colour of the granules varying from brown to red. On maturing the granules aggregate together forming a spherical mass. Now, in presence of moisture, the sporangium bursts liberating the zoospores. The zoospores are present in numbers in the sporangia, and it has been observed that they are discharged by these sporangial heads in consequence of simple pressure on the cover glasses. These zoospores are biciliated; they are spherical and measure 7-9 × 4-5·5μ, and are most conveniently stained with Romanowsky's stain, the cilia becoming blue and the body of the spore red.

The propagation of the alga takes place by means of these zoospores being carried by rain and air, and under suitable circumstances each of these zoospores can give rise to an algal growth. Mann and Hutchinson carried out some very interesting experiments with the view of ascertaining whether the leaf form alga could give rise to the stemform, and they succeeded in inoculating tea-stems by keeping them in contact with infested tea leaves and artificially supplying them with moisture. Further experiments should be carried out in this direction.

Check on the spread of the disease:—

Mann and Hutchinson, studying this question in connection with tea plants, say that the cause exists in preventing excessive reproduction. The time, during which the mature sporangia are capable of reproducing zoospores, is very limited, amounting as a rule only to a few days. After this they become dark-coloured as visible to the naked eye and then cease to respond to the addition of water. Hence it is obvious that unless rain, dew or watering of some sort occurs during the short period when the sporangia are capable of producing zoospores, their capacity for disseminating the disease is lost. The spread of the disease is also checked by the difficulty of getting the algal patches wetted and so not allowing the emergence of the zoospores during the time when these are in a suitable condition. Again, excessive moisture leads to the production of a very few reproductive organs and sometimes of none and thus limits the spread of the disease during the height of the rainy season.

Remedial measures for the disease have not been studied; but it is intended to attack the problem during the ensuing season. The treatment of the blight as it occurs on tea plants is fully described by Mann and Hutchinson. (Memoirs of the Dept. of Agriculture in India, Vol. I, No. 6, Bot. Ser. 1907).

The disease, as it occurs on mango trees, remains further to be studied, and the senior writer of this paper intends to study it thoroughly during the coming season, when he hopes to be able to clear up some of the more obscure points in the life-history of the alga and the mode of the spread of the disease specially on the mango tree.

EXPLANATION OF PLATE I.

A. A leaf of *Mangifera indica*, Linn., infested with *Cephaleuros virescens*, Kunze,—natural size.

B. A twig of the mango tree showing two points of attack by the alga,—natural size.

C. The thallus of the alga bearing the sporangia × 440.

EXPLANATION OF PLATE II.

A. A transverse section through the portion of a leaf of a mango tree infested with the alga showing the penetration of the alga through the leaf-tissue. The shaded portions are of brownish colour, × 420.

B. A thallus showing the ejection of zoospores from the sporangial head, × 440.

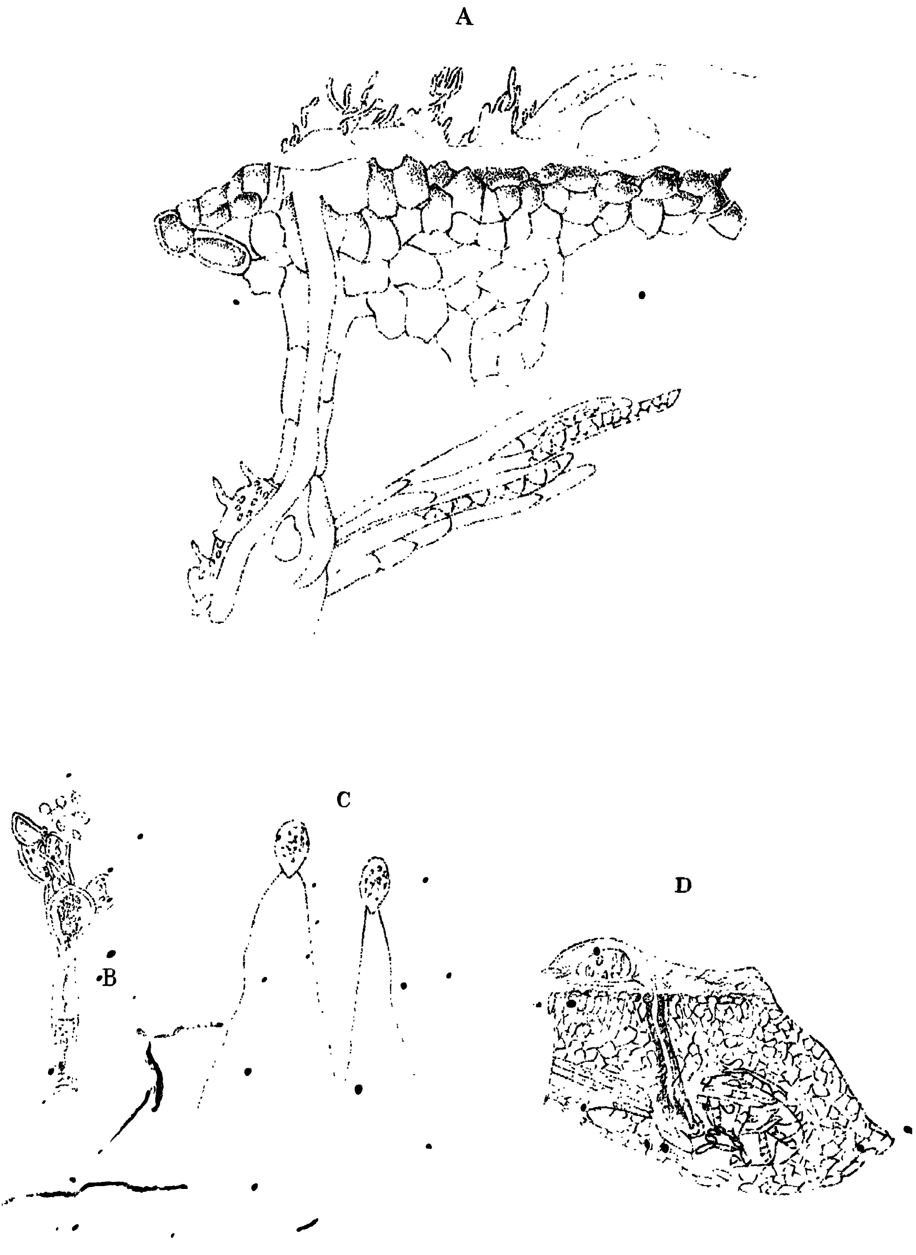
C. Two zoospores, × 1250.

D. A transverse section of a mango leaf showing the penetration of the alga through the leaf-tissue, × 130.

PLATE II.

B 1.

Chowdhury.



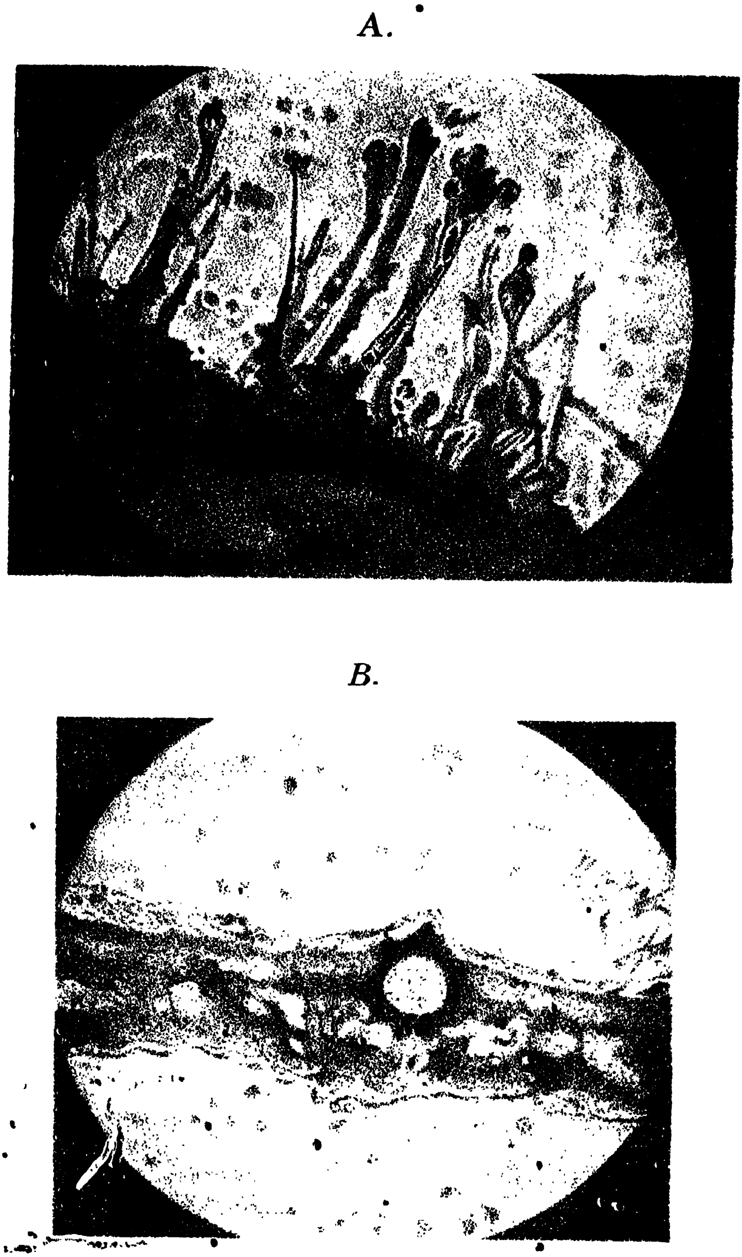
CEPHALEUROS VIRESCENS, Kunze., on MANGIFERA INDICA, Linn.

EXPLANATION OF PLATE III.

A. A microphotograph showing the thallus of the alga bearing the sporangia, × 250.

B. A microphotograph showing the fructifications of the lichen forming fungus on the mango leaf, × 250.

PLATE III.



CEPHALEUROS VIRESCENS, Kunze, l.c

MANGIFERA INDICA, Linn.

FORMATION OF WHITE GARNET-ROCK AS THE END-PRODUCT OF THE SERIES OF CHANGES INITIATED BY SAUSSURITISATION.

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INTRODUCTION.

The material of this paper is furnished by the study of a series of mineralogical changes observed in a remarkable assemblage of snow-white boulders of saussurite with associated scapolite and pure white garnet (grossularite) masses, all met with in a stream-bed in the Kokar-nag nullah, Islamabad District of Kashmir. These boulders are all closely inter-related and are presumably derived from the same source, though they have not been traced to their parent rock *in situ*. The changes observed in them belong to the class of mineral transformation commonly designated sassuritisation, but there are observable in the present case, phases of alteration both preceding sassuritisation as well as subsequent thereto.

The above-mentioned assemblage of boulders is composed of three distinct units:—(1) Saussurite, in snow-white rounded masses of the size of cocoanuts, containing small patches of serpentinised diallage; (2) Scapolite, in small well-rounded masses with large phenocrysts of uralite; and (3) Grossularite, in transluscent milk-white sub-angular pieces.

These boulders were picked up by a student mistaking them for white jade.

SECTION I.

The Saussurite Masses.

In its physical characters, the most typical of the saussurite masses is composed of a pure white opaque and dense minutely granular aggregate. Its specific gravity is 2·96; hardness between 6 and 7; it is very tough and compact, with a splintery fracture like jade. It has a dull subdued lustre on some parts which passes into chalky appearance at other places. It is capable of taking a good polish. Though opaque for the most part, it is transluscent in some edges. The substance is easily fusible to a white blebby glass in some patches, the remaining parts being infusible. It yields 2 to 3 per cent of water on strong ignition; is very slightly soluble in acids. In composition it is a silicate of alumina and lime with a small quantity of soda and the above percentage of water. There is no iron or magnesia or potash. In its chemical as well as optical characters the substance closely approaches zoisite. The optical characters and micro-structure are described later.

Scapolite-uralite Masses.

The associated scapolite pebbles are for the most part brilliantly white in colour, opaque, with a dull semi-vitreous lustre. Specific gravity, 2·90. The mineral has a hardness of 6–6·5; it is brittle, with a fairly perfect cleavage in one direction, transparent only in the thinnest sections. Fusibility 4. In its chemical composition it is a silicate of alumina and lime with a small percentage of soda. There is no chlorine, MgO or K_2O. In its composition, as in the above-named physical characters, it is an end-member of the scapolite group, meionite, being less silicious, more calcareous, denser, and more highly refringent and birefringent than the other scapolites grading towards marialite at the other end. In a thin section of this scapolite, examined under the microscope, the meionite base is found to be very intimately mixed with zoisite, and this association is of such a nature as to suggest the passage of the one into the other. In all micro-sections of the scapolite mass this change to zoisite is clearly visible as will be described presently.

Grossularite.

But the most interesting and noteworthy substance in this assemblage is pure white lime-garnet in translusecent sub-angular masses. It contains pale grey patches of much altered diallage or bastite. The specific gravity is 3·50. Hardness 7·5; fracture uneven, cleavage absent; white to colourless; translusecent; streak white; very tough, compact and homogeneous, very easily fusible to a clear white glass. Insoluble in acids, but it is decomposed on fusion with separation of gelatinous silica. In chemical composition it is a silicate of alumina and lime with a greater percentage of lime and less of alumina than in the two substances described above. It has a high R.I., about 1·75, with marked relief; it is perfectly clear and transparent and absolutely isotropic. It has a granulitic structure. In some sections the garnet is seen to be largely associated with granular and fibrous zoisite or epidote (clinozoisite); in fact in one boulder, with sp. gr. 3·38, the garnet and epidote are seen in about equal proportions in a state of intimate granulitic intergrowth.

SECTION II.

Micro-sections:

Saussurite.

Sections of all the three kinds of boulders were prepared, both of the typical-looking substance as well as of the transitional parts, with a view to observe the passage of one mineral into the other. To take the typical unaltered saussurite first: it is a colourless extremely fine-grained, almost homogeneous zoisite aggregate composed of fibres and granules. It is opaque except in the thinnest sections. The granulitic surface appears rugged due to the high relief. It is cut up by cracks and veins, the latter being filled with fibres of zoisite. There also appear some irregular indistinct bundles of fibres and rods of zoisite. No albite crystals or glassy grains of any other felspars can be distinguished, nor any vestige of the original lime-felspear. There is, however, a large amount of finely divided meionite as well as fairly coarse grains, wisps and plates of the same mineral present, which can be readily picked up by means of their low relief and smooth appearance. The interference colours of the main portion of the field are very low, blue-greys of the first order, the admixed scapolite showing straw-yellow. No distinct crystal outlines can be distinguished though there are many coarse grains and fibrous patches which extinguish parallel to their length.

Scapolite-zoisite.

A section of the scapolite mass, even of the freshest-looking part, shows the beginning of considerable alterations. In the clear and colourless plates of meionite, water-clear, roughly circular granules, with dark borders, appear in great number along the cleavage lines. In the more altered part the granules become more numerous and larger, and together with parallel-disposed bundles of zoisite fibres wholly replace the scapolite. The crystalline structure of the latter is lost and a confused crypto-crystalline aggregate is substituted, resembling the one described above under saussurite. The change proceeds from the cleavage and other cracks inwards. Under crossed nicols patches of the unaltered meionite are seen polarising in pale yellow tints, enclosed in a framework of zoisite aggregate of blue-grey interference colours and with a surface of high relief.

Grossularite.

In its most typical part, the grossularite-mass is a clear colourless and transparent granular mineral with but few inclusions or any accessory product. In other parts shreds and islets of a perfectly clear and transparent mineral with a fibrous structure and high refractive index and double refraction appear. The latter appear to be a nonferriferus epidote (clinozoisite); these become more abundant and occupy more than $\frac{2}{3}$ of some sections in jagged irregularly outlined granules dispersed at random in a clear granular matrix of the garnet. This granulitic structure is most conspicuous. The garnet is absolutely isotropic under crossed nicols, all sections giving in a selenite field a perfectly neutral tint. There is no sign of anomalous double reflection or of any microlitic inclusions. In the less completely altered parts the garnet is quite subordinate to zoisite and epidote. This part of the mass is clearly an intermediate product between saussurite and garnet and furnishes the most important evidence bearing on the subject of this paper. The relations of the two minerals to one another clearly suggest the conversion of zoisite to garnet.

The large phenocrysts of serpentinised pyroxene which are found adhering to the garnet masses contain a few small enclosures of the original unaltered pyroxene, which reveal their origin.

SECTION III.

The nature and stages of the mineralogical transformations; (1) Scapolitisation, (2) Saussuritisation, (3) Garnetisation.

From the fortunate circumstance of the preservation of large crystals and patches of uralitised pyroxene in the mass of the saussurite-boulders it is easy to infer the derivation of the latter. The uralitised patches represent the original pyroxenic constituent of a coarse gabbro, the remaining mass the original large felspar phenocrysts of the gabbro. Such coarse gabbro is known under the name of *Euphotide* in the Alps. The zoisite aggregate has resulted by the process of mineral alteration commonly designated "saussuritisation." There appears, however, in the present case an antecedent stage to the saussuritisation of the felspar, namely, the scapolitisation of the felspar.

The term "saussurite" does not denote a definite or constant mineral compound and there is no perfect agreement in its definition by various authors. Teall mentions that the substance is not always precisely of the same nature in all cases.* Weinschenk defines it as "composed of a dense aggregate of the calcium-aluminium silicates clinozoisite and lime-garnet and finely divided acid plagioclase, usually albite." † Dana says, "Saussurite is rarely, if ever, a homogeneous compound. In its composition it often approaches zoisite, of which it has been regarded as a soda-bearing variety" ‡

Saussuritisation is one of the usual modes of alteration of the basic plutonic rocks by the agents of contact metamorphism; the basic rocks being the most sensitive to this mode of alteration. The saussuritisation of the plutonic rocks finds its parallel in the extensive conversion of the basic volcanic rocks to green-stones and green-stone-schists.§

The scapolite mass described in I. and II. indicates that the first step in the alteration of the original gabbro was the scapolitisation of its lime-felspars. The conversion of anorthite into meionite is effected by the pneumatolytic action of the gases and vapours issuing from basic magmas, especially chlorine. This transformation proceeds as a deep-seated change in the zone of anamorphism (Van Hise). The change is one of the addition of lime to the anorthite molecule, accompanied by a slight increase in the density of the resulting compound.

In the micro-sections of scapolite the passage of the latter into zoisite is seen in all stages. The scapolite group of minerals is liable to extensive alterations, || and the production of zoisite or epidote as a secondary product from meionite, is not unusual. The two minerals are very analogous in composition, and the only change involved (when zoisite is concerned) is hydration, accompanied by some compactation of the original scapolite molecule.

The last step was the conversion of zoisite (saussurite) into grossularite Van Hise mentions that both meionite as well as zoisite have been found occurring as secondary products of grossularite. This derivation is natural since the latter mineral contains the elements in about the right proportion to form meionite or zoisite.¶ He gives chemical equations for the change, which is one of carbonation and

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* 'British Petrology,' pp. 149–152.

† 'Fundamental Principles of Petrology,' 1916, p. 132.

‡ 'System of Mineralogy,' 1916, p. 515.

§ Weinschenk, 'Fundamental Principles of Petrology,' p. 151.

|| Dana, 'System of Mineralogy,' p. 417,

¶ 'Treatise on Metamorphism,' p. 303.

hydration of the grossularite, in the zone of weathering. With the reversal of conditions, that is, the transfer of zoisite or meionite to the greater pressure and higher temperature prevailing in the deeper zones of the crust, it is probable that the equations are reversed and the zoisite molecule rearranges itself to the denser and more anhydrous grossularite molecule, which is more stable under the new environments. It thus appears that saussurite when subjected to further metamorphism results in grossularite, and that the production of a garnet-rock represents the ultimate stage in the series of alterations involved in the metamorphism of basic plutonic rock-masses.

Thus we can trace three distinct stages of alteration from anorthite to meionite, from meionite to zoisite and from zoisite to grossularite. This transition from felspar to garnet through the above stages is one of progressive increase of the specific gravities and of the relative chemical stability of the compounds involved, garnet being the densest and most resistant. This is in conformity with the view commonly held in petrogenesis, that the development of garnet is an important constructive process in the deeper zone of metamorphism.

Dr. L. L. Fermor in his paper on "Garnet as a Geological Barometer"* regards garnet as a measurer of earth-pressure. He has shown that the production of garnet is widespread in the zone of the crust lying below the level of the ordinary plutonic rocks, out of the common basic porotoxides and sesquioxides, that in the overlying zone of lesser pressure would result in such compounds as felspars, amphiboles, pyroxenes and olivines.

Weinschenk states that the production of garnet is the final stage in the series of alterations undergone by the basic igneous rocks. Where most completely altered by the agencies of contact metamorphism these basic rocks pass into eclogites, the felspars being recrystallised as garnet and the remaining constituents, together with the soda of the felspars, pass into omphacite or glaucophane. An earlier phase of the alteration is, according to this author, an amphibolite with saussuritised felspar and uralitised augite, while the most common intermediate stage is a garnet-amphibolite.†

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* Rec. Geol. Surv. of India, Vol. XLIII, 1913, pt. I, p. 41.

† 'Fundamental Principles of Petrology,' p. 132.

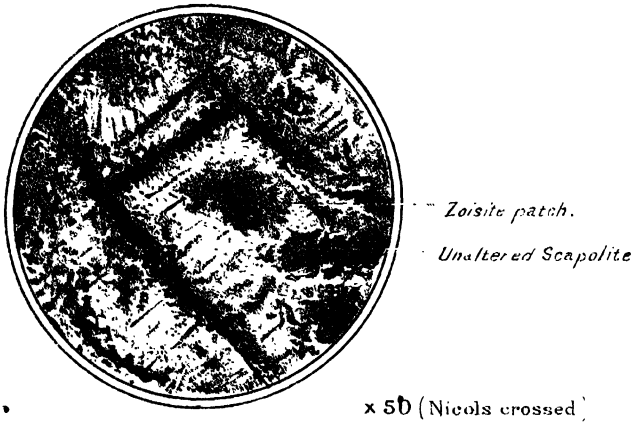

Fig. 1. x 50 (Nicols crossed)

SECTION OF SCAPOLITE Showing Alteration to Zoisite

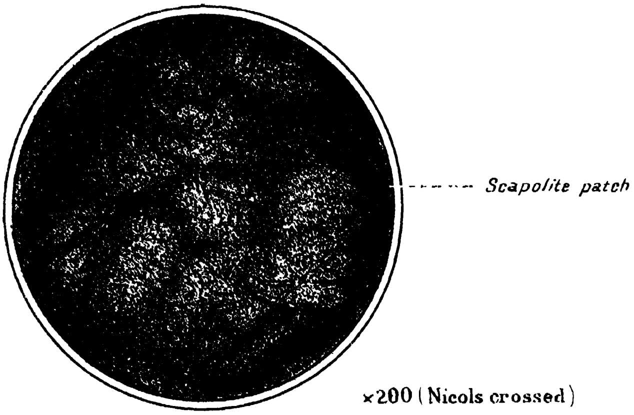

Fig. 2. ×200 (Nicols crossed)

SECTION OF SAUSSURITE BOULDER, Showing Confused Fibrous Aggregate Structure.

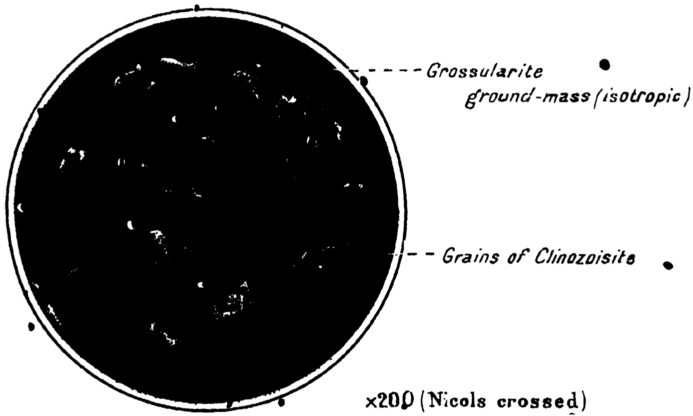

Fig. 3. ×200 (Nicols crossed)

SECTION OF GROSSULARITE MASS, Showing Inclusions of Clinozoiste. (*garnet absolutely isotropic*)

The above conclusions have been arrived at from physical, chemical and microscopic examination of 7 or 8 boulders, which constitute the sole material for the study. It must be stated here that these boulders have not been traced to their parent masses *in situ*. When that is accomplished, and when the field-evidence confirms the laboratory investigations, a very important and interesting point in petrology will be illuminated. In the meantime these conclusions are offered for what they are worth and because the material worked upon, *per se*, both by reason of its constitution and the remarkable nature of its association, has an interesting story to tell.*

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* A massive white eclogite-like rock, in many respects identical with the garnet-rock, was found by Mr. C. S. Middlemiss, C.I.E., Superintendent, Kashmir Mineral Survey, in Dras (Ladakh Prov.). It consists principally of white transluscent grossularite with specks of green omphacite. The structure and composition of this rock suggest that it is a secondary metamorphic product, derived by processes analogous to those described above, from some basic plutonic mass.

INTERACTION OF THIOUREA WITH MONO-, DI-, AND TRI-CHLORACETIC ACIDS AND MONOCHLORACETIC ESTER.

Sir Prafulla Chandra Ray, Kt., C.I.E., D.Sc. (Edin.), Ph.D., F.C.S., *Sir Taraknath Palit Professor of Chemistry, University of Calcutta, and* Manik Lal Dey, B.Sc.

It has been shown by one of us * that in the cold, thiourea reacts with monochloracetic acid in acetone solution, with the formation of the hydrochloride of formamidine thiolacetic acid. If the reaction mixture be heated ψ-thiohydantoin hydrochloride is the only product obtained.† It was thought necessary to study the reaction of thiourea with other chlorinated acetic acids as also monochloracetic ester. The results obtained are interesting, and show how the reactivity of chlorinated acetic acids in this particular instance decreases with the increase in the number of substituting chlorine atoms. In this connection it is necessary to bear in mind the well-known fact that the acidity of chlorinated acetic acids increases gradually as further chlorine atoms are introduced.

Although two different products are obtained in the reaction of thiourea with monochloracetic acid, according as the reaction takes place in the hot or in the cold acetone solution, with monochloracetic ester the same substance, ψ-thiohydantoin hydrochloride is formed whether the reaction mixture is heated or not, alcohol being eliminated. This fact has not been noticed by previous workers, although monochloroacetamide has been used to yield the same compound by a similar reaction.‡ Dichloracetic acid, on other hand is very slow to react. At ordinary temperatures the reaction is not complete before a week, and in fact the first crystals begin to appear after four days. The product is a mass of big diamond shaped crystals having a faint pink colour, probably due to some impurity. On analysis the substance is found to be the same compound, namely ψ-thiohydantoin hydrochloride. Evidently, during the first stage of the reaction monochloracetic acid is formed which interacts with thiourea to form the compound in the usual way thus:—

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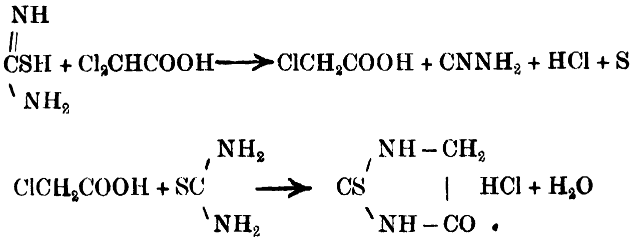
* *Trans. Chem. Soc.*, 105, 2159.

† Volhard, *Annalen*, 166, 385.

‡ Mulder, Ber., 8, 1264; Maly, Ber., 10, 1853.



$$\begin{array}{l}\mathrm{NH}\\ \|\\ \mathrm{CSH + Cl_2CHCOOH \longrightarrow ClCH_2COOH + CNNH_2 + HCl + S}\\ \diagdown\mathrm{NH_2}\end{array}$$

$$\mathrm{ClCH_2COOH + SC}\begin{array}{l}\diagup\mathrm{NH_2}\\ \\ \diagdown\mathrm{NH_2}\end{array} \longrightarrow \mathrm{CS}\begin{array}{l}\diagup\mathrm{NH-CH_2}\\ \qquad\ \ |\\ \diagdown\mathrm{NH-CO}\end{array}\mathrm{HCl + H_2O}$$

In aqueous solution Dixon obtained the same reaction.*

With trichloracetic acid, however, thiourea does not at all react at ordinary temperatures in acetone solution. The two substances, when mixed together, crystallise out side by side, with a slight decomposition. By carrying out the reaction in aqueous solution Dixon obtained the trichloracetic acid salt of thiourea.†

Experimental.

1. Thiourea and Monochloracetic acid.

Formamidine-thiolacetic acid hydrochloride. [NH_2. C (: NH). S. CH_2. COO H], HCl. The preparation was repeated and the substance obtained in a much purer form. Its properties already published‡ were confirmed. That it was a hydrochloride was further proved by the fact that its aqueous solution gave a precipitate of silver chloride with silver nitrate solution.

0·1936 gave 0·1591 AgCl and 0 2706 $BaSO_4$

Found Cl=20·33 ; S=19·20

Calc. for $C_3H_7O_2N_2SCl$, Cl=20·80 ; S=18·77

2. Thiourea and Monochloracetic ester.

Equimolecular proportions of thiourea and ethyl ester of monochloracetic acid, dissolved in acetone were mixed together in a flask which was left to itself overnight. Next day a crop of white crystals was found deposited at the bottom, together with a heavy oily liquid which on agitation solidified to a mass of white crystals. The crystals were collected, drained, washed with acetone and then recrystallised. The substance was found to be ψ-thiohydantoin hydrochloride. Its aqueous solution gave a precipitate of silver chloride with silver nitrate solution.

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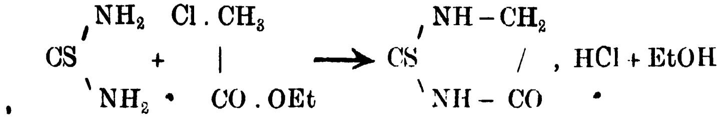
* *Trans. Chem. Soc.*, 1893, 816. † Ib. 1917, 688. ‡ Loc. cit.

In another preparation the mixture was heated on a water bath with a reflux condenser for twenty minutes. A heavy oil was at once formed which on cooling and stirring solidified to a mass of white crystals. They were identical with the former substance. The yield was quantitative.



$$CS\begin{cases}NH_2\\NH_2\end{cases} + \begin{array}{l}Cl.CH_2\\ \quad|\\ CO.OEt\end{array} \longrightarrow CS\begin{cases}NH-CH_2\\ \qquad /\\NH-CO\end{cases}, HCl + EtOH$$

0·1456 gave 0·1260 CO_2 and 0·0500 H_2O; found C=23·60 H=3·82

0·1968 gave 0 1803 AgCl; found Cl=22·66

Calc. for $C_5H_5ON_2SCl$, C=23·60; H=3·28; Cl=23·27.

3. Thiourea and Dichloracetic Acid.

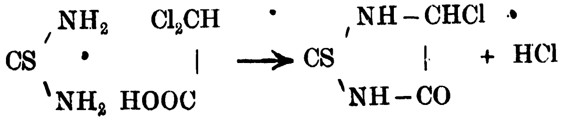
Acetone solutions of the two substances were mixed together in molecular proportions in a conical flask which was left to itself for several days, with its neck stuffed with cotton wool, so as to allow slow evaporation. After a week big crystals were found in a thick reddish mother liquor. The crystals were drained and washed with acetone and then with alcohol. They were diamond shaped and had a faint pink colour. They were insoluble in ether, alcohol and acetone but very soluble in water. The aqueous solution gave a precipitate of silver chloride with silver nitrate solution. From the mother liquor which contained various secondary products, free sulphur and thiourea were separated and identified Sulphur was separated by addition of acetone or alcohol to the mother liquor. It was filtered and recrystallised from carbon disulphide. Impure thiourea was obtained by evaporation of the liquid left after removal of sulphur.

0·1332 gave 0 1260 AgCl and 0·2102 $BaSO_4$,

Found Cl=23·40; S=21·67.

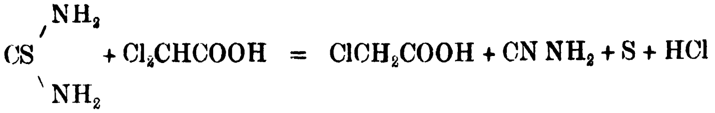
Calc. for $C_3H_5ON_2SCl$, Cl=23·27; S=20·98.

The substance is evidently ψ-thiohydantoin hydrochloride. At first sight it appears that dichloracetic acid should yield chlorothiohydantoin according to the following equation:—

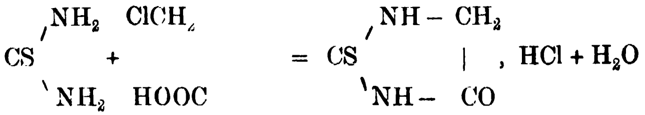


$$CS\begin{cases}NH_2\\NH_2\end{cases} \begin{array}{l}Cl_2CH\\ \quad|\\ HOOC\end{array} \longrightarrow CS\begin{cases}NH-CHCl\\ \qquad |\\NH-CO\end{cases} + HCl$$

But what evidently takes place is represented thus:—



$$CS\begin{matrix}NH_2\\NH_2\end{matrix} + Cl_2CHCOOH = ClCH_2COOH + CN\,NH_2 + S + HCl$$



$$CS\begin{matrix}NH_2\\NH_2\end{matrix} + \begin{matrix}ClCH_2\\HOOC\end{matrix} = CS\begin{matrix}NH-CH_2\\|\\NH-CO\end{matrix}, HCl + H_2O$$

By simple analysis it is very difficult to distinguish between the two possible products, since they differ only by two hydrogen atoms. But the presence of bye-products in the reaction mixture, as also the presence of chlorine ions in the aqueous solution of the substance favours the theory of formation of ψ-thiohydantoin hydrochloride. The electrical conductivity of this substance and that obtained by the interaction of *mono*chloracetic acid and thiourea, at same dilution, is practically the same, namely, $\mu_{100}=251$ and 256 respectively at 25°c, μ_{100} of pure hydrochloric acid at the same temperature is 400. Since thiohydantoin hydrochloride is a salt of the ammonium chloride type, its conductivity should be like that of ammonium chloride for which $\mu_{100}=140$. The experimental data points to the view that the compound is more or less completely hydrolysed into ψ-thiohydantoin and hydrochloric acid.

4. Thiourea and Trichloracetic Acid.

Equimolecular proportions of thiourea and trichloracetic acid were separately dissolved in acetone and the two solutions mixed together, and kept aside for several days at room temperature. After ten days a mass of white crystals was found lying in a thick mother liquor. The crystals were drained, washed with acetone and alcohol and then dried. On analysis the substance was found to be almost pure thiourea. It melted at 171°, the melting point of thiourea being 172°.

0·2218 gave 0·6743 $BaSO_4$. Found $S=41·76$

CH_4N_2S requires $S=42·10$

The mother liquor was found to be strongly acid, giving all the tests for trichloracetic acid. In fact the two substances did not react at all but the one crystallised out while the other being very deliquescent remained as a thick mother liquor.

In aqueous solution, however, Dixon* found a compound which he considered to be the trichloracetic acid salt of thiourea. We have repeated his experiment and confirmed his experiences.

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* Loc. cit.

ON THE OCCURRENCE OF FRANCOLITE IN STONY METEORITES.

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In two of his recent communications Merrill described the existence of a minor constituent in many aerolites.* The substance was found to be a calcium-phosphate and was identified with *francolite* because, when examined under the microscope, the constituent appeared to be biaxial. Merrill also enumerated the meteoric stones in which the constituent was found to occur. In the Geological department of the Presidency College, Calcutta, there is a collection of thin sections of meteorites suitable for microscopic study and it includes a few of the meteorites listed by Merrill and in most of them the mineral referred to by Merrill could be detected. In one of the slides of meteorites in the possession of the Geological Survey of India which I was permitted to examine I succeeded in detecting an apatitic constituent. From a study of these sections I have been led to form an opinion about the nature of this substance, which is somewhat different from that of Merrill and that for reasons given below.

The lime-bearing nature of the substance has been determined by producing a crop of gypsum and a deposit produced by ammonium molybdate has been attributed to its phosphatic nature. These qualitative determinations together with the quantitative analysis of the Alfianello and the Waconda stone indicate the qualitative chemical nature of the substance without any doubt. Farrington compiled a list of the published analyses of aerolites,† and a reference to this list shows that many of these analyses require revision as there are cases where the calcium-phosphate was detected microscopically, but the analysis did not show the presence of P_2O_5.

Francolite with which mineral the meteoritic phosphate was found to be identical had been originally described by Henry.[*] Subsequent analyses of *francolite* obtained from Cornwall were published in 1871 by Maskelyne and Flight and in course of the investigation of the Cornwall specimen CO_2 was also detected in the mineral.[†] A short description of the mineral was published by Lacroix [‡] and he was followed by Schaller.[§] The chemical formula obtained by Henry for *francolite* was $CaFl + 3(3CaO. PO_5)$ with a partial replacement of the lime by the protoxides of iron and manganese and though the failure to detect CO_2 detracts from the accuracy of the formula, the ratio of CaO to P_2O_5 is practically 3 : 1. The formula given for the Cornwall mineral is $\left.\frac{5\,[Ca_3\,(PO_4)_2\,]}{CaCO_3}\right\} + 2CaF_2$ which shows that the ratio of CaO to P_2O_5 is 3 : 1. According to Lacroix the formula for the mineral is $[(PO_4)_8\, Ca_8\, (CaF)_2,\, CaCO_3,\, H_2O]$ which gives also 3 : 1 as the ratio existing between CaO and P_2O_5.[||] According to Schaller the average ratio of P_2O_5 to CaO is 3 : 10·44 and the simplest formula developed for the mineral is $9CaO.3P_2O_5.CaF_2.CO_2.H_2O$.

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* *Proc. Nat. Acad. Sci.*, U.S.A., Vol. I (1915), pp. 302-307; *Amer Journ. Sci.* Vol. XLIII (1917), pp. 322-4.

† *Field Mus. Nat. Hist.*, publ. 151, Geol. Ser., Vol. III, No. 9 (1911), pp. 195-229.

From what has been stated above it appears that broadly speaking the average ratio of CaO to P_2O_5 in *francolite*, is very nearly the same as exists in a typical *apatite* and for a mineral to be identified with *francolite*, this ratio should not differ very much from 3 : 1. According to Merrill, the Alfianello stone was analysed by Dr. Whitfield with the result that 0·344% of CaO and 0·08% of P_2O_5 were obtained and from this analysis the ratio of CaO to P_2O_5 was calculated to be 9 to 2.[¶] This statement is open to correction as will appear below :—

	Percentage	Molecular weight.	
CaO	0·344	÷56 ..	·0061
P_2O_5	0·08	÷142 ..	.00056

From these figures it is quite clear that the ratio of CaO to P_2O_5 is 11 to 1 and not 9 to 2. It may thus be concluded that, if the results of Dr. Whitefield are correct, the mineral cannot be identified with *francolite* or with any mineral of an apatitic type.*

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[*] *Phil. Mag.*, Vol. 36 (1850), pp. 134-5.

[†] *Journ. Chem. Soc.*, Vol. 24 (1871), pp. 3-5.

[‡] *C.R. Acad. Sci.*, Paris, Vol. 150 (1910), pp. 1213-7.

[§] *U.S. Geol. Surv. Bull.*, No. 509 (1912), pp. 91 ff.

[||] As Schaller says, according to Lacroix, the ratio of P_2O_5 to CaO is 3 : 11. (op. cit., p. 99).

[¶] *Proc. Nat. Acad. Sci.*, (1915), p. 304.

A careful consideration of the whole question has led me to think that no great reliance can be put on Dr. Whitefield's results for finding out the chemical composition of the phosphatic mineral. The Alfianelo stone was analysed by Foullon in 1883 and phosphoric acid was detected by him though not *apatite*. The analyst records †: —

" Phosphorsäure wurde qualitative in so geringer Menge nachweisen, dass auf eine quantitative Abscheidung verzichtet wurde. Sie stammt wohl von dem Phosphorgehalte sehr kleiner Quantität von Schreibersit her, welche Annahme durch den Umstand wesentlich unterstützt wird, dass auch in dem ausgezogenen Eisen Phosphorsäure nachweisbar ist."

Foullon's description shows that the stone consists of *olivine*, *bronzite* and *maskelynite* all of which contain CaO. Of these *olivine* " ist an seiner lichten Farbe, die ihn durchsetzenden Klufte und durch die Zerstezbarkeit durch Salzsäure kenntlich ".‡ About *maskelynite* it has been observed that " die chemische Zusammensetzung stimmt mit keinem bekannten tesseralen Mineral; sie hat aber Ähnlichkeit mit der eines Labradorites von Labrador ".§ From these facts it appears to me more probable that the amount of CaO estimated by Whitfield was due not only to the calcium-phosphatic mineral of the stone, but also partially to *olivine* and *maskelynite*, and hence the ratio of CaO to P_2O_5 is so great as 11 to 1. It may be mentioned further that Foullon's analysis does not show the presence of CO_2 which is an essential constituent of *francolite* and by which it can be distinguished from apatite.

Apatite had long been recognised as a doubtful constituent of meteorites.|| Berwerth described the presence of this mineral in the Kodaikanal meteorite ¶ while Ludwig and Tschermak obtained it in

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* The possibility that the above statement might have been due to some typographical mistake was not overlooked. The calculation can only be uphold if the p.c. stands for the ratio between the percentage and the molecular weight. But 0·344 and 0·08 as such a ratio for CaO and P_2O_5 respectively can be obtained from 19·26 p.c. (=0·344 × 56) of CaO and 11·36 p.c. (=0·08 × 142) of P_2O_5. These values are not possible either on the assumption that the whole stone was analysed or that the phosphatic constituent was separated and analysed. The values would be too high for the former (the more probable) and too low for the latter assumption.

† *Sitzb. d. kais. Akad. d. Wiss. Wien.*, LXXXVIII, pt. 1 (1884), p. 433. This stone was also analysed by P. Maisson who detected phosphorus in it (Farrigton, op. cit., p. 227).

‡ Loc. cit., p. 436. § *Sitzb. d. Kiss. Akad. d. Wiss. Wien.*, LXV (1872), p. 130.

|| *Journ. Geol.*, IX (1901), p. 530. ¶ *Min. u. Petr. Mitth.*, XXV (1906), p. 188.

the aerolite from Angra dos Reis.* A chemical analysis of the Angra dos Reis meteorite shows that it contains 0·13% of P_2O_5 and 0·17% of CaO, quantities which are quite consistent with the ratio of 1 : 3 as existing between P_2O_5 and CaO. In the Waconda stone 0·26% of P_2O_5 was obtained † and this would require 0·30% of CaO for *apatite*. This stone was also analysed by Smith‡ who detected the presence of phosphorus and lime in very small quantities which were not estimated. If the ratio obtained from Whitfield's analysis be correct, then the amount of CaO corresponding to 0·26% of P_2O_5 would have been 1·12% and such a large proportion was sure of having been estimated.

The arguments derived from the chemical characters on which the phosphatic mineral has been identified with *francolite* being open to doubt, it now remains to consider the physical characters on which the mineral has been supposed to represent *francolite*. The first impression that is produced by a microscopical examination of the constituent is that of *apatite* and, according to Merrill, the substance is to be distinguished from *apatite* chiefly by its biaxial nature which was established on an examination of the thin sections of the stones. In course of a study of the sections available to me I succeeded in getting some dark bars, but it appears to me that nothing decisive can be settled on the available evidence and Merrill's description with the figures is also not very convincing. Students of petrology will agree with me when I say that only on rare and fortunate occasions one may come across a basal section of a uniaxial mineral in a rock slice, as of quartz in sections of granite and hence the following statement of Merrill does not go much to establish the biaxial nature of the mineral :—

"The possibility of the apparent biaxial interference figures being those of a uniaxial mineral cut parallel with an optic axis was considered, but deemed wholly improbable from the fact that in not one of the many sections examined was I able to find a uniaxial figure. It seems improbable that among so large a number should not be found at least one, did such exist."§ It may be mentioned here that biaxial apatites are also known.‖

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* Ibid., XXVIII (1909), p. 110

† *Amer. Journ. Sci.*, 1917, p. 323

‡ Ibid., 1877, p. 212.

§ *Amer. Journ. Sci.*, (1917), p. 3 4, footnote. Those remarks show that Merrill himself is not very sure about the biaxial nature of the substance.

‖ Iddings, *Rock Minerals*, p. 523.

It appears from what has been stated above that the constituent referred to by Merrill may be *apatite* though further verifications are necessary for establishing the point beyond any doubt. As observed by Merrill, the microscopic examination shows that this phosphatic mineral is a product of the last stage of consolidation as quartz in granite. When we remember that the evidence of the chondri shows that the order of the relative abundance of the chondritic minerals is that of their fusibility,* considerable doubt is thrown on the identity of the meteoric calcium-phosphate with normal apatite.

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* *Journn. Geol.*, IX (1901), p. 176.

TEMPERATURE COEFFICIENTS OF PHYSIOLOGICAL PROCESSES.

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In this article it is proposed to subject to critical examination the results obtained with regards to the effect of temperature on physiological processes. Before proceeding to the consideration of these reactions I shall briefly state the results obtained in the case of purely chemical reactions and then try to show how far these relations are applicable to physiological changes.

In homogeneous medium the following general results have been obtained :—

(*a*) The higher the order of the reaction, the smaller is the coefficient of temperature, in other words, unimolecular reactions have higher temperature coefficients than polymolecular reactions under identical conditions.

(*b*) The greater the velocity of a reaction the smaller is the temperature coefficient

(*c*) The temperature coefficient of a positively catalysed reaction is smaller than that of the uncatalysed reaction and the greater the concentration of the catalyst the greater is the fall in the temperature coefficient.

In the case of negative catalysis, a reaction which is catalysed (negatively), has a higher temperature coefficient than the uncatalysed reaction. In this case, the greater the concentration of the catalyst the greater is the increase in the temperature coefficient.

In the case of heterogeneous reactions, the following points have been established :—

(*a*) Diffusion is the guiding factor in the velocity of heterogeneous reactions.

(*b*) With heterogeneous catalysts which cause reaction between the substance in question to take place with practically infinite velocity, the actual rate of reaction will be determined solely with which the substance is diffused to the surface of the catalyst.

(c) If the heterogeneous velocity is that of the diffusion process, one will always get a unimolecular coefficient for the reaction in question, independent of the actual order of the more rapid chemical reaction, which accompanies the diffusion process. Hence it is useless to try and to determine the order of a heterogeneous reaction, from the velocity with which they proceed.

(d) The temperature coefficients of heterogeneous reactions are small (viz. about 1·2 for a 10° rise).

In this connection it is interesting to note that photochemical reactions have small temperature coefficients (viz. about 1·1 for a 10° rise).

Now I shall discuss the results obtained in physiological processes with regard to the influence of temperature on them.

The relation between the temperature and the velocity of respiration has been studied during the last few years both for plants and animals. The principal object of these investigations has been to find out whether respiration can be considered as a chemical process.

From the researches of Clausen,* Blackman,† Kuijper,‡ Lehenbauer,§ Miss Leitsch,‖ Miss Saunders (private communication) and others we find that the temperature coefficients of plant processes generally lie between 2 and 3 for a 10° rise of temperature.

Brown and Worley ¶ have shown that the temperature coefficient of the velocity of absorption of water by different seeds is about 2 for a 10° rise. If the values of the velocity coefficients are calculated from their results, we see that they follow the unimolecular formula.

The researches of Veley and Waller ** show that the Arrhenius formula can be applied to the influence of temperature on the velocity of the action of drugs on muscles.

Very large number of experiments have been made on the influence of temperature upon metabolism both in cold-blooded and in warm-blooded animals. But comparatively few of them have been made under standard conditions. In most cases animals have been free to move about and even in cases where they have been tied muscular movements have not been prevented or muscular tone abolished. In these conditions a fundamental difference has been observed between the effects of temperature upon cold-blooded and upon warm-blooded animals. In cold-blooded animals the respiratory exchange almost always rises with increasing temperature, but generally irregularly and to a very different degree in different animals.

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* *Landwirt. Jahrbuch* . XIX, 1890.

† *Annals of Botany*, 1905, XIX, 281.

‡ *Rec. Trav. Pot. Neerl.*, 1910, VII, 131.

§ *Physiological Researches*, No. 5, August, 1914.

‖ *Annals of Botany*, January 1916.

¶ *Proc. Roy. Soc.*, 1912, 85 B, 546.

** *Proc. Roy. Soc.*, 1910, 82 B.

In the case of bees Marie Parhon * finds that the temperature in the cluster of bees inside the hives shows a very striking constancy throughout the year.

In intact warm-blooded animals, a fall in the surrounding temperature regularly causes an increase in the respiratory exchange—thanks to the mechanism of "chemical heat regulation."

In all the experiments so far mentioned both on cold-blooded and on warm blooded animals we have to do with two distinct effects of temperature, viz. one upon the central nervous system causing variation in the innervation of different organs and especially of the muscles and one upon the tissues themselves influencing the reaction velocity of the metabolic processes.

In the warm-blooded animals the action of low temperature on the skin produces reflexly innervation of the muscles resulting either in movements or in increase of tone.

In the cold-blooded animals the processes in the central nervous system itself are probably acted upon, and increased muscular activity is produced by increasing temperature except in the cluster of bees which in the aggregate reacts against the temperature somewhat after the fashion of a warm-blooded animal.

When the influence of temperature on the metabolic process is to be studied the nervous influence must be excluded and the experiments must be made under standard conditions.

It has been found repeatedly both on man and on animals that even a slight increase in body temperature over the normal produces an increase in the standard metabolism.

It follows from the experiments of Krogh† and others that the velocity of catabolic reactions increases in all animals with rising temperature up to a maximum at and above which temperature has deleterious effect upon the organism. The maximum temperature probably differs considerably for different animals, but very few determinations have been made so far.

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* *Ann. des Sc. Nat. Zoo.* Ser. 9, 9, 1–58.

† *Biochem. Zeit.*, 1914, LXII. 266.

The more rigorously standard conditions are maintained the more regular is the influence of temperature observed.

Chick and Martin[*] find that the coagulation of hæmoglobin by heat has the temperature coefficient 13·8 for the elevation of 10° whilst in the case of albumen it is higher. In this connection it is interesting to note that Von Schrœder[†] has found that a solution of gelatine has a viscosity of 13·76 at 21°c. and 1·42 at 31°c, i.e. about 10, times less with an elevation of 10°.

The results obtained by Chick and Martin show that the temperature coefficient of coagulation of proteins by water is an exceedingly high one compared with effect of temperature on most chemical reactions. In the majority of instances the reaction velocity is increased about 1·1 times for 1°c, i.e. 2 to 3 times for a rise of temperature of 10°. Even the biological processes of germination of seeds, respiration of plants and growth of bacteria fall within this range.

On the other hand many reactions in which complex protein bodies are concerned have been shown to possess high temperature coefficients which are comparable with those obtained for heat coagulation. The destruction of emulsion by heat has according to Tammann[‡] a temperature coefficient of about 7·14 for a 10° rise between 60° and 70°. Bayliss found that the action of trypsin to be hastened 5-3 times for some germs in accordance with a logarithmic law. Ballnev (1902) found the disinfection of anthrax spores by steam to take place from 9 to 11 times more quickly by raising the temperature 10° and the law of Arrhenius is equally applicable to his results.

Chick and Martin[§] have shown that the disinfection of vegetative forms of bacteria with phenol and other coal tar derivatives has a temperature coefficient of 8 to 10 for a 10° rise of temperature.

On the other hand the disinfection by silver nitrate and mercuric chloride has a much lower coefficient and that is about 2.

The high temperature coefficient for the coagulation of egg albumen has a counterpart in that for the velocity of destruction by hot water of the haemolysins, in the vibriolysins, tetanolysin and goat serum.

Madsen and his colleagues found the influence of temperature to be in accordance with the law of Arrhenius and the velocity of this reaction to be doubled if the temperature were raised 1°C. They also showed that the action of hot water upon some agglutinins to be similarly influenced by temperature.

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[*] *Jour. of Physiol.*, XLV, 40.

[†] *Zeit. Phys. Chem.*, 1903, XLV, 75.

[‡] *Zeit. Phys. Chem.*, 1895.

[§] Loc. cit.

This marked influence of temperature is extremely useful for men and animals. When a toxin enters the system, the temperature of the body rises by two or three degrees and we get the phenomenon of fever and the poison is destroyed about 10 or 20 times more quickly at this fever temperature.

Hartridge * finds the temperature coefficient for heat coagulation to be as great as 726 for a 10° rise for some protein matter. In this connection it is interesting to note that the decomposition of sulphur trioxide by heat has 419 for its temperature coefficient for a 10° rise at about 30°.

Watson † applying Ostwald's isolation method to Miss Chick's results finds that in the disinfection of certain bacteria with phenol, the molecules (N) of phenol reacting with those of the bacterial constituent are in the proportion of 5·5 to 1. As regards the metallic salts the same law holds good for disinfection by silver nitrate and the molecules (N) of silver nitrate reacting with those of bacterial constituents are in the proportion of 1 : 1. In the case of mercuric chloride, however, the above relation between the concentration of disinfectant and the average velocity of disinfection is maintained only if the former is expressed in terms of the corresponding concentration of mercuric ions· Under these circumstances. (N) has the value 4·9 for anthrax spores and 3·8 for *B* paratyphosus. But the temperature coefficient of the disinfection by phenol is very high though the reaction is approximately heptamolecular. On the other hand, in the case of silver nitrate the reaction is approximately bimolecular and the temperature coefficient is small, viz. 2 for a 10° rise These results are contrary to our experience in ordinary chemical reactions where the greater the order of a reaction the smaller is the coefficient of temperature.

Kanitz,‡ Synder,§ Cohen Stuart,|| Pütter,¶ and others have tried to represent the influence of temperature on physiological processes by the rule of Van 'tHoff, but it is not very important whether the tempera-

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* *Jour. of Physiol.*, 1912, Vol. XLIV, 34.

† *Jour. Hygiene*, 1908, 8, 536. ‡ *Temperatur und Lebensvorgange*, 1915.

§ *Amer. Jour. of Physiol.*, XXII, 1908, 309.

|| *Proc. k. Akad. Wetsch. Amsterdam*, 1912, XX, 1270.

¶ *Zeit. Allg. Physiol.*, 1914, XVI, 617.

ture coefficient has the value 2 or 3, the important point to establish is whether the formula of Arrhenius * or the formula of Harcourt and Esson † which is applicable to ordinary chemical reactions is also applicable to physiological processes.

Blackman ‡ has accepted the validity of the Van 'tHoff rule and has found the value 2·1 between 9° and 19°. He has assumed that this value of the temperature coefficient remains constant at higher temperatures; this assumption is contrary to our experience in ordinary chemical reactions, the temperature coefficient for a 10° rise becomes smaller as the temperature rises. This falling off of the temperature coefficient with increase of temperature is also expected from the Arrhenius formula. Evidently the conclusions of Blackman would have been more correct had he accepted the Arrhenius formula.

Looking at the whole problem from a broad point of view it seems that temperature has two effects on vital processes :— (*a*) the increase of the velocity of the chemical reaction involved in the physiological changes; (*b*) the destruction of the living cells.

At low temperatures the first effect is predominant since the harmful effect does not begin to play its part.

Thus the problem for us is to investigate the effect of temperature on vital processes at low temperatures that is, before the harmful effect on the living cells has begun and we shall probably see the same quantitative laws which are applicable in the domain of ordinary chemical reactions *in vitro*, are also applicable to vital processes taking place in nature.

Enzymes and Colloids reign supreme in life processes and the Brownian movement of these particles does away with the diffusion layer characteristic of heterogeneous reactions and makes them analogus to positively catalysed reactions taking place in homogeneous medium and hence we expect to find the same laws governing both ordinary chemical reactions and life processes.§

In conclusion I suggest that it is desirable to study the problem of acclimatization scientifically from the point of view of the influence of temperature on life processes.

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* *Zeit. Phys. Chem.*, 1889, IV, 226.

† *Phil. Trans.*, Series A, Vol. 186, 817 (1895); Vol. 212, 187 (1912).

‡ *Annals of Botany*, 1905, XIX, 281.

§ Compare (Dhar, *Proc. Akad*, Vetnsk. 1919).

Summary :—

(*a*) Physiological processes take place mostly in heterogeneous medium. The Brownian movement of the colloidal particles present in the reacting substances does away with the diffusion layer characteristic of heterogeneous reactions and makes the physiological reactions similar to positively catalysed reactions taking place in homogeneous medium. Consequently the temperature coefficients of physiological processes instead of being small (viz. about 1·2) are generally greater than 2 for a 10° rise.

(*b*) The spontaneous destruction of certain toxins is highly influenced by temperature and this fact is extremely useful to the human body because in the phenomenon of fever the poison is killed very rapidly.

(*c*) Before the destructive effect of temperature begins to set in, the Arrhenius formula connecting temperature and velocity is generally applicable to physiological processes.

SPORE-CULTURE OF *PANÆOLUS CYANSCENS*,
B. AND BR.

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Panæolus cyanacescens is common in Bengal, usually appearing during the rainy season from June to August. It is usually found on heaps of dung. It has a central stalk, 2½ to 3 inches long, which turns deep-blue when lightly touched; the cap is umbrella-shaped and coloured ashy white; the central umbo is reddish-yellow and the diameter of the cap about 1½ inches. The hymenial surface is black with a bluish tinge. Any part of the plant when bruised turns deep blue. The spores are distinctly black, oval and tapering at both ends; the cystidia are very prominent, their white tips projecting beyond the general surface. A more detailed description has been given in a paper contributed to the Journal of the Asiatic Society of Bengal which will come out very shortly (1921.)

The specimen of *Panæolus cyancescens* was gathered in the compound of the Carmichael Medical College, Belgachia, on the 15th of August 1919. The method of pure culture followed was that described in Bulletin No. 85 of the Bureau of Plant Industry.

A petri-dish was filled to about two-thirds with a mixture of ¾ horse dung and ¼ cow-dung and was stirred up with a certain amount of water; it was completely sterilized in a Koch's sterilizer at 100° C for 75 minutes each day for three successive days. For the purposes of inoculation a healthy specimen was chosen, washed with soap water, then kept in a glass dish previously washed with alcohol. The hymenium of the specimen was cut into small pieces with a sterilized pair of scalpels; four of these pieces were placed each in one of the four quadrants of the sterilized dung-medium, contained in the petri-dish, just below the surface of the medium, taking all possible precautions to prevent contamination from foreign bodies. After puting on the lid of the petri-dish, the latter was kept under a bell-jar at the ordinary room temperature, which ranged from 76° to 84·5° F. on the day of inoculation, namely the 15th August, 1919. No appreciable change took place till the 25th August, that is the 10th day, when the first growth of a mycolium in the shape of a delicate felt-like structure was observed. The temperature on that day ranged from 76° to 86·5° F. Fig. I of the accompanying plate shows the mycelial growth. Three days later, a thin white stalk surmounted by a small cap, which had grown during the previous night of the 28th August, pressed against the lid of the petri-dish, the slender stalk having become curved due to its rapid elongation. The lid having been taken off, the fungus went on growing under the bell-jar at the ordinary room temperature The following measurements of the stalk from 28th August onwards were taken :—

On 28th August the stalk measured 1 inch, temp. 74° to 79·6° F.
On 29th ,, ,, ,, 2½ inchs, ,, 73° to 83°F.
On 30th ,, ,, ,, 3½ inches, ,, 73·5° to 85° F.

The cap unfolded and became umbrella-shaped on the 31st August, the stalk measured 4 inches, the temperature ranging from 74·5° to 85·5° F.

The full size was reached on the 31st of August, the stalk having become slightly longer and the cap somewhat smaller than in the normal form. Fig. II of the accompanying plate shows the fungus on the 14th day of its growth. The method of tissue-culture as described above, has been highly successful in America in producing spawn and has led to a considerable development of the mushroom industry.

Following this tissue-culture method, spore-germination of local edible varieties of Agaricaceæ, viz. *Volvaria terastia*, B. and Br., *Lepiota albuminosa*, Berk., *Lepiota mastoideus*, Fr. and *Agaricus campestris*. Linn. is now being attempted in the laboratory, using the dung medium. The results of the experiments will be communicated on a later date. It is thus hoped to produce spawn of indigenous edible mushrooms, based on such laboratory pure culture methods, so as to make the industry independent of the importation of foreign spawn most of which has proved unreliable, evidently owing to the spawn having perished before reaching this country and thus giving no response in the plains of Bengal. Mushroom growing may in this way become a special industry in India, as it has been in several parts of Europe, and in America as late as 1913 and 1914.

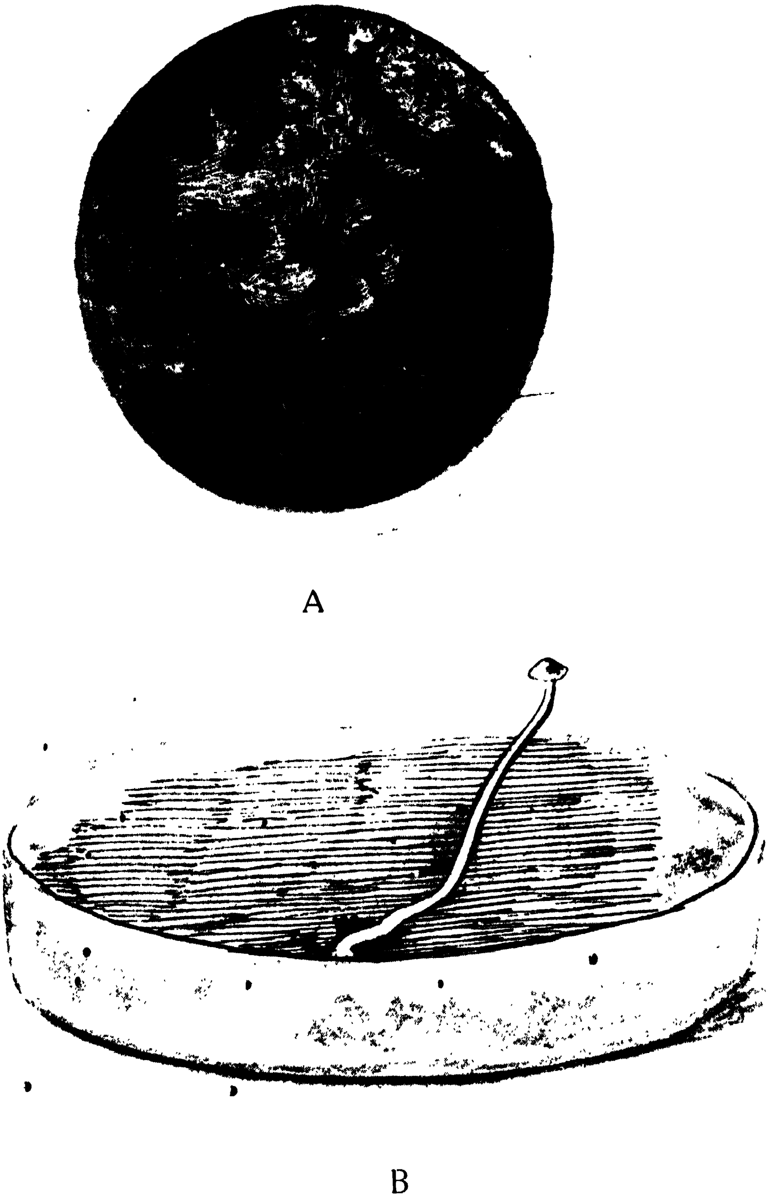
EXPLANATION OF PLATE.

A. Mycelial growth of *Panæolus cyanascens*, B. and Br. Growth on the 10th day from inoculation.—reduced to ⅔rd of the original.

a. Felty mycelium.

b. Dung medium.

B. *Panæolus cyanascens*. B. and Br. Growth on the 14th day from inoculation. (stem 2½″), -reduced to ⅔ of the original.



PANÆOLUS CYANASCENS.

ON INDIAN FOSSIL PLANTS AND THE GONDWANA CONTINENT.*

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Specimens of fossil plants from India were described as early as 1822, but little work was done upon the Indian plant bearing rocks until 1856, when Dr. Thomas Oldham published the first classification of the plant-bearing beds. The work of examination was vigorously carried on by W. T. Blanford, T. W. Hughes, W. King, O. Feistmantel and others; this work extended over a period from 1859 to about 1886, after which date, our knowledge of the plant-beds of India has been advanced mainly by work in the extra-peninsular parts such as the Himalaya. The great bulk of the collection of fossil plants in the custody of the Geological Survey of India dates from about 1860 to 1885 and is thus over 30 years old.

The oldest era into which the geological time is divided is called *Agnotozoic* because there are no fossils in the rocks of this oldest division, and consequently we know nothing of the life of those times. (ἀγνωτός, unknown and ζωή, life.)

In the Cambrian period, the earliest known recognisable fossils occur, and they all belong to marine forms of life. There are no plant remains, with the exception of some very doubtful sea-weeds. In the next succeeding period, we find fossils of calcareous sea-weeds, but the remains of land plants are poorly preserved, and of doubtful relationships. One of these doubtful plants has been found at the base of the Silurian in Spiti.

It is the earliest known Indian plant and is believed by some to have been a sea-weed, by others to be a land plant. It is not possible to estimate the age of the Silurian rocks in terms of years, but recent work on the radioactive minerals found in these rocks enables us to guess at the age roughly, and it is now believed that the Silurian period was in existence something like 450 million years ago.

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A series of experiments made on rocks of upper Carboniferous age seem to show that the close of the Carboniferous took place about 350 million years ago. It is therefore practically impossible for the human mind to realise the enormous antiquity of the fossil plants which I hope to deal with in this paper.

The oldest known land plants of India were found by Dr. Hayden in Spiti about 15 years ago. They are of Lower Carboniferous age, and are interesting because they belong to genera which were cosmopolitan and which are found widely scattered from Europe to Australia, the Spiti plants are very fragmentary fossils, but one of the best specimens is that of a fern named *Rhacopteris*.

It is a remarkable fact that in Lower Carboniferous times we find that the flora was a comparatively uniform one all over the whole world. Collections have been made from Greenland, Ireland, South Africa, South America and many other widely separated localities. In all these areas we meet with the same genera, and it has been thought that such uniformity of flora implies a fairly uniform climate all over the world at that epoch, a climate mild and moist, permitting the abundant growth of vegetation. Let us try to imagine the nature of the vegetation of those remote times.

The Lower Carboniferous trees belonged mainly to two classes of vascular cryptogams, the *Equisetales* and the *Lycopodiales*, classes which are now represented by small plants, and which occupy quite a humble position in the botanical world.

The first class *Equisetales* is represented in the present day by the single genus *Equisetum* or the Horsetail. These plants are found in moist or boggy places.

In this plant the leaves are very rudimentary and the branches occur in whorls separated by rather long internodes. The fossil Equisetales of Lower Carboniferous times are for the most part fairly large trees, which owing to their reed-like appearance have been given the generic name of *Calamites* (from Latin *calamus*, a reed).

The second class, that of the *Lycopodiales* or Club-Mosses, is also a small and unimportant group in the present day. They are plants covered with a velvet of small scale-like leaves, which are borne on long branching stems, except in the case of the aquatic genus *Isoetes* or the Quill-wort, which has a stunted stem and long leaves.

None of the present day *Lycopods* are trees, but the largest is a species of *Selaginella* which is found in Borneo, growing to two or three feet in height. It is surprising therefore to find that in Carboniferous times this class was represented all over the world by numerous species including many large trees.

The most well-known genus is called *Lepidodendron*. The *Lepidodendrons* were of gigantic size. A few years ago a fossil tree-trunk was found near Bolton in Lancashire. This measured 114 ft. in length up to the point where it commenced to branch. Prof Seward remarks that " A fully grown *Lepidodendron* must have been an impressive tree, probably of sombre colour, relieved by an encircling felt of green needles on the young pendulous twigs. " The leaf-bases are arranged upon the stem of *Lepidodendron* in rows, giving a very characteristic pattern to the surface of the stem.

Until recent years it was believed that the great majority of the smaller Carboniferous plants were *ferns*. Laterly, however, it has been recognised that very many of these fern like plants were not really ferns at all but bore *seeds*, and thus resembled in some ways the still living group of Cycads. In fact they were a much more advanced and more highly developed class than we originally thought. They are now known as *Pteridospermae* or Seed-ferns.

The above is only an outline sketch of the vegetation of Lower Carboniferous times, but it is as well to remember that as far as we know the vegetation was very uniform all over the world from Australia to Greenland.

In Upper Carboniferous times, a very extraordinary event occurred, which profoundly affected the distribution of vegetation all over the earth, and separated the flora of the world into two-well marked botanical provinces, one in the northern hemisphere, exclusive of India, and the second in the southern hemisphere and India. A sudden change of climate took place, which seems to have caused intense cold in the southern hemisphere and India, while the northern hemisphere appears not to have been affected in any marked degree. A glacial boulder clay of Upper Carboniferous age has been found in various parts of India, the Godaveri Valley, South Rewah, Bengal, Western Rajputana, the Salt Range, but it does not appear to have been formed in Kashmir or Spiti. Ice-scratched boulders are quite common, and the boulder clay has in one instance been observed to rest upon an old rock surface, which is polished, scratched and grooved by the action of the ice. This is in the Penganga River, S. W. of Chanda in the Central Provinces. The remarkable discovery of a glacial period in Upper Carboniferous times, as evidenced by ice-scratched boulders and boulder clay was first made by Messrs. Blanford and Theobald in the Talchir Coal-field near Cuttack, and was published in 1856. The progress of the Geological Survey revealed the fact that all the great coal-fields of Bengal and Peninsular India, show this glacial boulder bed present at the base of the coal-bearing series of rocks. The glacial boulder bed rests upon pre-Cambrian or Archean rocks, and it is itself the basal bed of a series of sandstones and clays which appear to have been deposited by rivers, and which contain abundant fossil plants and bones of animals which lived in rivers of marshes. There are no marine fossils in these beds, except in deposits which are situated near the present coast line, in which a certain intermixture of the land and marine fossils has taken place.

This series of plant bearing sandstones and clays covers a period of geological time extending from the upper Carboniferous to the lower Cretaceous. It was called the *Gondwana* system by Dr. Feistmantel. It has been separated into two divisions, an upper and a lower. and these in turn into groups, each group being divided into smaller divisions called stages. We possess many specimens of plants from each of these stages with the exception of the Maleri stage.

It has been already mentioned that there is evidence that ice covered many parts of the Indian peninsula and that it extended as far north as the Salt Range, but that it probably did not reach Spiti or Kashmir. Messrs. R. D. Oldham, La Touche and Middlemiss, basing their opinions on a study of the kinds of rock of which the boulders of the glacial boulder bed in the Salt Range were composed, concluded that these boulders must have been transported as glacial moraine from the Aravali Hills in Rajputana. Dr. Fermor came to a somewhat similar conclusion as to the origin of the boulders of the boulder-bed in South Rewah. He believed that the boulders had been carried from the N.W. If reliance is to be placed upon these theories we must believe that the Aravali Mountains were a centre of dispersion.

South of Chanda, in the Penganga valley, the grooves and scratches upon the rock surface underlying the boulder-bed show that the ice travelled from S.W. or S.S.W., that is from the direction of the Nilgiri Hills, which may have been another centre of dispersion.

But India is by no means the only country which affords evidence of upper Carboniferous glaciation. Boulder beds of glacial origin have been found in deposits of this age in Australia and Tasmania, in South Africa, and quite recently have been discovered in the Falkland Islands, and in Brazil. In Australia the evidence of glaciation is found widely spread, in Queensland, S. Australia, New South Wales' W. Australia and Tasmania. The centre of dispersion is believed to have been a point to the south of Adelaide in a region now occupied by the open sea.

In South Africa a glacial boulder bed known as the Dwyka Conglomerate has been found over a great extent of country throughout Cape Colony, Natal, the Transvaal and the Orange River Colony. Here again we have numerous ice-scratched boulders and grooved and striated rock surfaces are found on the rocks upon which the conglomerate rests. The centre of dispersion of the South African ice was situated to the North of the Transvaal, and there appears to have been one huge ice-sheet, since the grooves are always directed from North to South. The Dwyka Conglomerate itself resembles the well-known boulder clay of England, which last is of course very much more recent in origin.

Just as in India the Talchir Boulder Bed is the base of the Gondwana System, so also in South Africa the Dwyka Boulder bed is the base of a series of plant bearing beds, the exact counter part of the Gondwanas, and known as the Karoo System. The Karoo System is famous on account of its well preserved fossils of land animals, mostly reptiles and amphibians, and with them some mammal like animals, which are of peculiar interest.

Thus in all the continents south of the equator (excepting the unexplored Antarctica) we have evidence of glacial conditions at the close of the Carboniferous. This glaciation extended in the northern hemisphere to Malaya and India. There is however no sign of it in Europe or in North America or Asia On the contrary many geologists are of opinion that a warm tropical climate prevailed in the northern hemisphere at that epoch. This opinion is based on an examination of the structure of the arborescent *Lycopods* or *Lepidodendrons* already mentioned.

These huge trees were very abundant in Europe and other parts of the northern hemisphere in the Upper Carboniferous. They show in their structure special adaptation of the woody fibres to assist a rapid ascent of sap, and they are protected against a rapid evaporation by a thick corky bark clothing the stem. We find the same features in the Calamites. Moreover the leaf-scars on the trunks are separated by very irregular intervals; this is believed to indicate a rapid increase of growth. Lastly there is a complete absence in the stems of the tree Lycopods of well-defined rings of growth, such as are to be found in trees growing in a climate subject to seasonal variation of winter and summer. Such rings of growth are however found in wood of Lower Gondwana age from India, the Cape, and Australia.

The problem of this upper Carboniferous glaciation becomes more perplexing, the further we investigate it. No satisfactory explanation has yet been given of the cause of the ice-age. Prof. Koken of Tübingen in 1907 produced a wonderful map of the world as it was in this epoch, showing a distribution of land and sea quite different from that of the present day.

He supposes that India, Africa and South America were united together to form one great continent; this opinion is not original but is adopted from Dr. W. T. Blanford, and is based upon very strong reasons, which I will afterwards mention. Most geologists believe that Australia was also attached by an isthmus of land perhaps by way of New Guinea and Sumatra to India in those times, but Prof. Koken has not taken this view, and in this he seems to be wrong. Anyhow this great southern continent was named Gondwanaland by E. Suess, and all over it we find glacial boulder beds at the base and above them strata containing a very remarkable flora the plants of which were quite different from those of the northern continents.

But the main new idea which Prof. Koken introduced in this extraordinary map of his was that the south pole was then in a different position from that which it occupies now. From the astronomical point of view there seem to be formidable objections to such a theory, but still it might have been accepted as a possible theory, if it had explained the facts that it was intended to explain. By putting the south pole in the middle of the Indian Ocean, Prof. Koken thought he could explain to us how South Africa, India and Australia were then covered with ice Unfortunately the very next year after the publication of Koken's paper Dr. I. C. White of the Brazilian Coal Commission proved the existence of glacial boulder clay of Upper Carboniferous age in Brazil—a locality very close to Koken's equator. Moreover an examination of the direction of the glacial striae and grooves has shown that the ice did not radiate out from any point in the Indian ocean as

Koken supposed. Koken's explanation is difficult to accept at the present day. Others have supposed that the earth's orbit was then very much more eccentric than now, and that the southern winter took place when the earth was in aphelion. This would result in a continual cooling of the southern hemisphere and a corresponding warming up of the northern. But on this theory we cannot comprehend how glaciation could affect northern countries like India, unless indeed the climate of India was affected by cold currents coming from the south, just as Labrador is cooled by an Arctic current in the present day. Finally Prof. Svante Arrhenius has suggested that a diminution of the amount of carbonic acid gas present in the air would cause a considerably greater loss of radiant heat - a loss which would increase rapidly with the increase of altitude. On this theory a small elevation of the land above sea level would suffice to explain the glaciation of Gondwanaland. But here again the explanation is not satisfactory, because in the Salt Range, glaciated boulders are found mixed up with marine deposits and marine fossils, and we are almost forced to conclude that here at least the ice must have been in existence at sea level.

In short the upper Carboniferous ice-age is an unsolved problem which bids fair to puzzle scientists for many years to come.

In all countries which have been affected by this glaciation, the beds above the boulder clay contain coal-seams and fossil plants. The first coal seam is usually in close proximity to the glacial boulder bed. At Vereeniging for instance, south of Johannesburg, the first seam is only about 8 yards above the Dwyka Conglomerate. Almost immediately after the final retreat of the South African glacier, conditions favourable to vegetation re-established themselves. The plants found immediately above the glacial bed are very different from those of the same age in the northern hemisphere. Owing to the abundance of two plants known as *Glossopteris* and *Gangamopteris*, this flora has received the name of the *Glossopteris*-flora.

These two genera, *Glossopteris* and *Gangamopteris* are very closely allied, the distinction between the two being that there is a midrib, in *Glossopteris* which is absent in *Gangamopteris*. We do not know whether these plants belong to the class of Ferns or whether they are members of the *Pteridospermae* or Seed-Ferns which I mentioned in the beginning of my paper. We often find seeds in the same deposits as these plants but we do not know to what plant they belonged. Until 1897 we did not know what sort of stem or rhizome the *Glossopteris* leaf grew upon. Fossil stems had been known under the name of *Vertebraria* from the Gondwana rocks, but they had been referred to the class of *Equisetales*. In 1897 Mr. R. D. Oldham found some *Glossopteris* leaves attached to a *Vertebraria* stem, and thus disclosed the fact that the fossils we had been calling *Glossopteris* and *Vertebraria* were part of the same plant.

Now both *Glossopteris* and *Gangamopteris* are abundantly found all over Gondwanaland. More remarkable still, the species of these plants are the same throughout widely separated areas. Thus *Glossopteris browniana* occurs in Brazil, South Africa, Madagascar, India, and Australia, while *Gangamopteris cyclopteroides* is found in South America, South Africa and India.

There are a number of other genera and species which are found in this *Glossopteris*-flora. I should mention however that a second and not less important characteristic of the *Glossopteris*-flora of the Gondwanaland continents is the absence of those huge Lycopods, the *Lepidodendrons* and their allies, which are so abundant in and so characteristic of the flora of the northern continents. Not a single Lycopod is known from the Gondwana System of India * nor from Australia, although in this latter country they grew in the period preceding the upper Carboniferous glaciation. Only in two places is an intermixture known of the northern Lycopod-flora with the southern *Glossopteris*-flora. One of these is in South America at San Juan in the Argentine and elsewhere, and the second is in the Transvaal where near Vereeniging *Lepidodendron* has been found with *Glossopteris*. It appears probable that the giant Lycopod, being tropical trees were practically exterminated in the southern hemisphere by the Upper Carboniferous glaciation. It is therefore easy to understand their absence. But we do not yet know from what source the *Glossopteris*-flora came and what caused it to spread suddenly over Gondwanaland, immediately after the ice-age. Dr. W. T. Blanford many years ago expressed the view that the Antarctic Continent was the original area of development of many of these plants, and Prof. Seward of Cambridge more recently wrote :— " there is not wanting evidence in favour of the *Glossopteris*-flora having been first differentiated in an Antarctic continent towards the close of the Carboniferous epoch."

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* Recently, however, Messrs Servard and Sahni have described a doubtful specimen of *Bothrodendron* from the Giridih coal-field.

Since the present Antarctic continent is completely barren, except for a little moss and lichen, it was necessary to find some evidence of a milder climate in former ages, before Blanford's theory could be accepted.

On Feb. 2nd, 1912, specimens of *Glossopteris* were found by Dr. Wilson who accompanied Capt. Scott on his last expedition to the South Pole. The specimens were found on Buckley Island which rises from the great Beardmore Glacier in Latitude 85°, to a height of 8,000 ft.

The words of Capt. Scott's diary read as follows:—

"We lunched at 2, well down towards Mt. Buckley, the wind half a gale and everybody very cold and cheerless. However better things were to follow, the moraine was obviously so interesting that when we had advanced some miles and got out of the wind, I decided to camp and spend the rest of the day geologising.

We found ourselves under perpendicular walls of Beacon Sandstone, weathering rapidly and carrying veritable coal-seams. From the last, Wilson with his sharp eyes, has picked several pieces of coal with beautifully traced leaves in layers."

These leaves turned out to be *Glossopteris indica*; a plant which is also known to occur in the Gondwana beds of India.

In the oldest group of the Gondwanas, the Talchir Group, the flora is a remarkably scanty one, and is in its purest state. Later on a certain amount of intermixture and reciprocal migration took place between the northern and southern botanical province. But the Talchir Group is distinguished by the absence of many species which are abundant in the succeeding group of Damudas. One of the Talchir species is *Neuropteridium validum.* It is believed to have been a seed-fern, and is common in Brazil, Argentine, the Transvaal, and India.

Another Talchir species, *Cordaites hislopi*, belonged to a family known as the *Cordaiteae*, which became extinct at the close of the Rhaotic stage of the Triassic Period, that is the Parsera stage of the Gondwana system. They were large trees attaining 100 ft. in height, and in appearance somewhat like the *Conifers*, especially the Kauri pine of New Zealand. Their nearest living representative is the Maiden-Hair tree of Japan or the *Ginkgo*.

In the next group, the Damudas we find abundant plant fossils amongst which *Glossopteris* and *Gangamopteris* predominate. A very characteristic genus of the Damudas is *Sphenophyllum.*

The *Sphenophylls* are placed by palaeobotanists in a separate class, but they have some characteristics in common with the *Equisetales* or Horsetails. They became extinct at the close of the Permian and are not found in rocks of later age than the Damudas. They were slender graceful plants: their stems were less than half an inch in diameter. The stems are jointed and ribbed and bore whorls of leaves usually six in number.

Another common plant of the Damudas, which is also found in the Panchet Group above is *Schizoneura.* This is one of the *Equisetales* or Horsetail class. The long straight parallel nervation of the leaves and the ribbed jointed stem is very characteristic.

Several frond-genera of northern affinities appear in the Damudas and probably indicate a migration into Gondwanaland from the north.

In the lower Permian epoch this reciprocal migration of genera is undoubted. *Glossopteris* spread into Eastern Europe into the northern botanical province, and is found together with northern species on the Dvina River in Western Russia.

After the close of the Damuda epoch and the beginning of the Panchet epoch, the distinction between the two botanical provinces is much less marked. We find in the Panchet group several animals which are common to India and Europe. Some of the plants also are common to Europe and India. The Panchet group is recognised by the absence of *Gangamopteris* and *Sphenophyllum* which had died out at the close of the Damuda epoch. *Glossopteris*, *Cordaites*, and *Schizoneura* are still abundant in the Panchet group, but they die out at the close of this epoch and are not found in the next group. Perhaps the most beautiful species of the Panchet group is a fern named *Danaeopsis hughesi.* This species has been found in South Africa, China, Tonkin, and India, and appears to characterise the Rhaetic or Parsora stage.

Another plant, probably a seed fern *Thinnfeldia odontopteroides* is found in the Rhaetic or Parsora stage in Australia, South Africa, India, South America, and in various European localities.

The Talchir, Damuda, and Panchet flora consisted mainly of ferns, seed-ferns, giant horsetails or *Equisetales*, and the *Cordaitales*, which were allied to the *Conifers.* There were few *Cycadophyta,* and of course no flowering plants or trees, i.e *Angiosperms.* In the next succeeding groups, we find a remarkable advance in the flora. Ferns and seed-ferns are still fairly common, but the most abundant of all are the *Cycadophytes.* The *Conifers* are beginning to become common in the Mahadeva group, and are abundant in the Jabalpur-Umia group.

The Upper Gondwana vegetation thus differed widely from that of the Lower Gondwanas. At the commencement of the period true ferns and seed-ferns were abundant. *Conifers* often closely resemble recent types, and the family now represented by the *Gingko* Maiden-Hair tree of Japan was of considerable importance. The *Cycadophytes* were however as characteristic of the Upper Gondwana flora as the flowering plants are of that of to-day.

The *Cycads* of to-day are a small family—a mere remnant of what was once a great botanical group. They now include only nine genera, of which the type genus *Cycas* is found in India and Australia, two are found only in Australia, two in South Africa and four in America. They bear cones but their leaves resemble those of ferns. The sexes are always upon different plants, both the male and female plant bear cones. These are often of a beautiful appearance: in one South African genus, the scales of the cone are orange coloured and the seeds which show between are a bright scarlet.

The vast majority of mesozoic *Cycadophytes* were very much more highly organised than those of to-day. One of the most interesting families of mesozic cycadophytes has been named the *Bennettiteae.* In these plants the fructifications were borne laterally upon the stem and did not grow from the top of the stem as they do in the present day family. Another point of difference is that in the same fructification are found stamens bearing the pollen and in their midst the female apparatus. In fact we find an arrangement which can only be paralleled by our present-day flowering plants, in which a ring of stamens surround the pistil. In the centre is an ovuliferous cone. Surrounding this is a whorl of stamens, which do not in the least resemble the simple stamens of a modern flower, on the contrary they are very complex organs and closely recall the fertile fronds of a fern. Round the stamens there are several whorls of bracts which correspond to the sepals and petals of the modern flower, and which might be called a perianth.

Of course such delicate structures as inflorescences and fructifications are very seldom preserved in the fossil state. As a rule we have nothing except leaves and stems to go upon, and therefore it is impossible to say what leaves from India belong to the *Bennettiteae* and what to some less advanced family of *Cycadophytes*. But there is reason to suppose that the *Bennettiteae* were very abundant in India in mesozoic times. One genus *Williamsonia* is abundant in India, several species being the same as ones found in Yorkshire.

Williamsonia gigas, as restored by Dr. Scott, shows the rhomboidal leaf scars, the cycadean leaves and the fructifications which are rather like an artichoke in shape. In a specimen of *Williamsonia* from Mexico, the stamens were preserved in the fossil state, and were found to be simple and not pinnate fern-like fronds such as these of the more typical *Bennettiteae*. This occurrence of simple stamens is very interesting, because it seems to show a close connection between some of these advanced types of cycadophytes and our modern flowering plants. In fact the well-known palaeobotanist Saporta grouped these types of plants into a special class which he called *pre-Angiosperms*, regarding them as ancestral types to the Angiosperms or Flowering plants.

It is remarkable that not a single specimen of the group of *Angiosperms* or flowering plants has been found in the Upper Gondwanas. The last stage of the Gondwana System, the Umia stage, is of Lower Cretaceous age. In the Upper Cretaceous, *Angiosperms* are abundant in various parts of the world, and leaves of Birch, Beech, Oak, Walnut, Plane, Maple, Holly, and Ivy have been described from rocks of that age.

It is remarkable then that although these flowering plants are so abundant and well differentiated into families in the Upper Cretaceous, there are hardly any fossils of this group from the Lower Cretaceous and none in the Jurassic.

It is hard to explain this sudden appearance of the flowering plants upon the earth. The old school of geologists found a refuge from such difficulties in a pious belief in the docttines of divine creation, and the great naturalist Cuvier attributed such sudden changes to some great catastrophe which destroyed the former forms of life and enabled the Divine Creator of the world to re-people it with new and higher types of plants and animals. But in this modern and sceptical age, we are not satisfied with these simple beliefs, and scientists have sought to account for the sudden appearance of the flowering plants as due to the development of insect life and the supersession of wind-fertilisation by insect fertilisation. The flower was evolved for the purpose of attracting the insect.

In the Carboniferous period when the oldest Gondwana deposits were laid down, the insect life of the world consisted mainly of cockroaches and dragon flies. These last were extraordinarily large, one species having a body 14 inches long with a spread of wings of about two feet. In the Upper Jurassic period the bees, wasps, butterflies and moths begin to appear. Palaeobotanists of to-day believe therefore that it was from some group of the many highly advanced types of *Cycadophytes*, such as this family of *Bennettiteae*, the *Angiosperms* were evolved. This was due to the fact that a new method of fertilisation had become possible, that is fertilisation by means of insects which carry the pollen from flower to flower, as opposed to the old method of fertilisation by wind-blown pollen

As regards the distribution of species of plants in upper Gondwana times, it cannot be said that the flora may be divided into two provinces as was the case in Lower Gondwana times. On the contrary the flora appears to have been fairly uniform all over the world. It is remarkable that even in Graham Land, in the South Polar Regions, Jurassic plants flourished, several genera such as *Thinnfeldia* being the same as those of India. The re-establishment of a more uniform climate over the earth brought about a wide distribution of certain genera and species.

With the close of the Gondwana Period, the most interesting chapter of Indian Palaeobotany is ended. In the Tertiary Era silicified fossil wood both of monocotyledonous and dicotyledonous type abounds.

In parts of Burma, the whole surface of the ground is strewn with this fossil wood, as in the Yenangyat oil field.

A REVISION OF THE FAMILY OPHRYOSCOLECIDAE, CLAUS.

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Introduction.

The family Ophryoscolecidae was first established by Stein (21) in 1867 under the name of Ophryoscolecina for the reception of two genera of Infusoria, *Entodinium* and *Ophryoscolex*, both described by him in a previous paper (20). Kent (14) followed his classification in his *Manual of Infusoria.*

In 1888 Schuburg (17) split up the genus *Entodinium*, Stein, and established a third genus, *Diplodinium*, for what he considered to be two species of *Entodinium* of Stein; but the species of Schuberg were altogether new to science and were distinct from those of Stein, as shown by the later observers.

Bütschi (3) recognised these three genera as constituting the present family.

In, 1889 and 1890 Fiorentini (10, 11) described a large number of new species of *Entodinium*, Stein, and *Diplodinium*, Schuberg, and established a fourth genus, *Didesmis*, for a number of new forms.

In 1892 Schuberg (18) criticised the paper of Fiorentini, and showed that Fiorentini had described two already known species of *Ophryoscolex*, Stein, as new species of *Diplodinium*, Schuberg.

In 1895 Railliet (15) suggested some alterations in the nomenclature of two species of *Diplodinium*, Schuberg, described by Fiorentini. He considered that *Diplodinium denticulatum*, Fiorentini, was identical with the form described by Stein and named it *D. dentatum*, Stein, whereas he designated *D. dentatum*, Fiorentini, as *D. mammosum.*

The same year Bundle (4) established a new genus, *Cycloposthium*, for a species of *Entodinium*, Stein, described by Fiorentini, thus raising the number of genera to five.

The same year, Eberlein (9) reviewed most of the species already described by Stein, Schuberg and Fiorentini, and described one new species of *Ophryoscolex*, Stein, and one of *Diplodinium*, Schuberg.

The next year, Delage and Hérouard (7) gave accurate and concise diagnoses of the three genera, *Ophryoscolex*, Stein (p. 467), *Entodinium*, Stein (p. 468) and *Diplodinium*, Schuberg (p. 468), which may well form a trustworthy basis for their identification.

In 1903 Hickson (13) suggested the removal of the genera *Cycloposthium*, Bundle, and *Didesmis*, Fiorentini, from the present family to a new one of their own. He gave brief diagnoses of the three genera left in the present family.

In 1912 Brumpt and Joyeaux (2) founded a new genus, *Troglodytella*, thus raising the number again to four.

In 1914 Da Cunha (5) studied these parasitic infusoria in Brazil and made a brief, but careful survey of the complete literature on the subject, which has served to remove a good deal of confusion regarding the nomenclature and identification of several species of *Ophryoscolex*, Stein, and *Diplodinium*, Schuberg. Thus he removed several species of *Diplodinium*, Schuberg, described by Fiorentini to the genus, *Ophryoscolex*, Stein.

The same year Poche * established a new family *Cycloposthiidae* for the reception of two genera, *Cycloposthium*, Bundle, and *Didesmis*, Fiorentini, formerly placed in the present family.

The next year Da Cunha and Travassos (6) described several new species of *Entodinium*, Stein.

The same year Awerenzew and Mutafowa (1) described a new genus, *Metadinium*, and a new species of *Diplodinium* Schuberg, and several of *Ophryoscolex*, Stein. They criticized the view of Eberlein that *Diplodinium dentatum* and *D. denticulatum* of Fiorentini were identical, and considered *D. dentatum* of Eberlein (with 5 to 6 spines) to be synonymous with *D. denticulatum*, Fiorentini. They established a new species, *D. fiorentini* (with 3 spines) which they doubtfully considered to be identical with *D. dentatum*, Fiorentini.

The same year, Sharp (19) as a preliminary to his histological study of what he considered to be *Diplodinium caudatum*, Fior., discussed the validity in the position of some of the species of *Diplodinium*, Schuberg, as such. He further described a number of new varieties of *Diplodi-*

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* *Arch. Protistenk.* Bd. 30 (1913), p. 259.

nium ecaudatum, Fior. all of which however seem to be members of the genus *Ophryoscolex*, Stein. He further appended a table noting the differences in the character of the " dorsal membranelles zone " and in the number of the " vacuoles." Thus in *Ophryoscolex* Stein, the " dorsal membranelles zone " is an incomplete spiral encircling $\frac{4}{5}$ of the entire body, and the vacuoles are 5-6 in number In *Diplodinium*, Schuberg, the " dorsal membranelles zone " is " transverse, encircling less than $\frac{1}{2}$ of the entire body, and the vacuoles are 2 to 4 in number." Lastly in *Entodinium*, Stein " the " dorsal membranelles zone " is absent and the " vacuole " is single. This differenciation between *Ophryoscolex*, Stein and *Diplodinium*, Schuberg cannot be accepted in consideration of the diagnostic characters of the genera already established by previous workers.

In 1917 Reichenow (16) described a new species of *Troglodytella gorillae* and a new subspecies of *T. Abrassarti*, Brumpt and Joyeaux.

In 1918 Hasselmann established a new genus *Cunhaia* raising the number of genera to five. The paper is referred to by Da cunha (5A) ; but original reference is not available.

The present paper aims at arranging the genera and species of the Ophryoscolecidae according to the latest researches and describing a number of new forms from the rumen of the cow and goat.

The material has been received from the slaughter-house at Tangra through the kindness of the Health Officer, Corporation of Calcutta.

The method of collection well described by Sharp (19) has been closely followed, and has proved successful. The animalcules were examined fresh and after fixation with hot alcoholic corrosive solution in the fluid collected, without staining or staining with borax carmine and Delafield's haematoxylin. An electric table lamp of 100 candle power has been used for illumination, ordinary, daylight being quite insufficient for the purpose.

Fam. Ophryoscolecidae, Claus.

1859. Ophryoscolecina, Stein. *Lotos, Zeitschr f. Naturwiss.*, Prag., Vol.9, p. 58.
1874. Ophryoscolecidae, Claus. *Grundzüge der Zool.*, 3 Aufl., 1. Lief, p. 179.
1896. Ophryoscolecinae, Delage and Hérouard, *Traité de Zool. concrète*, T. 1, p. 468.
1903. Ophryoscolecina, Hickson, " Infusoria " in Lankester's *Treatise on Zool.*, p. 409.

I. Subfam. Ophryoscolecinea, subfam. nov.

A well-developed peristome with a spiral of membranelles ; with or without a secondary spiral of membranelles in the anterior end of the body.

Gen. 1. ENTODINIUM, Stein.

1858. Stein, *Abhandl. der. kgl. Boehmischen Ges. d. Wiss.*, Folge V, Bd. X, pp. 69–70.

1881-9. Bütschli, "Protozoa" in Bronn's *Thierreich*, p. 1738.

1888. Kent, *Manual of Infusoria*, p. 654.

1896. Delage and Hérouard, *Traite de Zool. Concrète*, 1, p. 468.

1903. Hickson, "Infusoria" in Lankester's *Treatise on Zool.*, p. 410.

1. E. BURSA, Stein.

1858. Stein, vide supra.

1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 264–268. Schuberg (17) considers this species to be identical with ***Diplodinium bursa***, described by himself. It is undoubtedly a different species from that of Schuberg.

2. E. MINIMUM, Schuberg.

1888. Schuberg *Zool. Jahrb. Syst., Abt.* Bd. III, p. 411.

1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 271.

1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 66.

3. E. NUDUM, sp. n. (Fig. 6).

Body greatly elongated and tapering at both ends. Anterior end narrow, truncate and wholly occupied by the peristome, which presents a wide fissure on one side. Posterior end tapering to a blunt rounded process. A stout elongated macronucleus A micronucleus at the side of the macronucleus. Two contractile vacuoles in the anterior half of the body. Lt. 0·05 mm. Greatest diameter 0·036 mm.

Found in the rumen of the goat.

The present species differs mainly from *E. bursa*, Stein, in general shape of the body, in the absence of a posterior depression and in the shape of the macronucleus. It differs from *E. minimum*, Schuberg, in general shape, in the number of contractile vacuoles and in the absence of a posterior anal slit.

4. E. MAMMILATUM, Da Cunha and Travassos.

1914. Da Cunha and Travassos, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 183.

5. E. OVATUM, sp. n. (Fig. 3).

Body elongated, tapering at both ends. Anterior end narrow and truncate, with a deep fissure on one side and entirely occupied by the peristome. Posterior end tapering to a blunt end with an oblique anal canal opening on one side. Ectosarc very thick at either end of the body. Macronucleus long and band-like. A single micronucleus in a depression of the macronucleus. Two contractile vacuoles. Lt. 0·055 mm. Greatest diameter 0·03 mm.

Found in the rumen of the goat.

6. E. SUBMAMILATUM, sp.n. (Fig. 7).

Body elongated, somewhat flattened laterally. Anterior end tapering narrow and truncate, and wholly occupied by the peristome. Posterior end with a rounded process on one side and a projecting curved blunt process on the other. Macronucleus narrow, elongated, somewhat constricted behind the middle and bent posteriorly. Micronucleus at the side of the constricted portion. Two contractile vacuoles on the side opposite to that of the macronucleus. Lt. 0·055 mm. Greatest diameter 0·032 mm.

Found in the rumen of the goat.

The present species differs from *E. mammillatum*, Da Cunha and Travassos, in general shape, in the shape of the posterior processes and macronucleus, and in the number of the contractile vacuoles.

7. E. ELONGATUM, sp.n. (Fig. 8).

Body elongated, somewhat flattened laterally and tapering at both ends. Anterior end narrow, truncate and wholly occupied by the peristome, which presents a narrow fissure, on one side. Posterior end broadly conical and pointed, and with a slightly curved rounded process on one side. Ectosarc greatly thickened on the side of the curved process. Macronucleus narrow, elongated and bandlike. Micronucleus at the side of the macronucleus near its anterior end. Two contractile vacuoles near the macronucleus. Lt. 0·065 mm. Greatest diameter 0·035 mm.

Found in the rumen of the goat.

This species so closely resembles the preceding one, that the two might be considered identical; but as the two differ from each other in general shape, in the shape of the macronucleus and in the presence of a thick ectosarc in the present one, and as they are not linked together by intermediate forms, they are best regarded as distinct, though closely related species.

8. E. ELLIPSOIDEUM, sp.n. (Fig. 2).

Body ellipsoid, slightly flattened from side to side. Anterior narrow end truncate and entirely occupied by the peristome. Posterior end forming a large rounded process with a finger-like process at the side and projecting beyond it. Macronucleus short, stout and bandlike. A single micronucleus. Two small contractile vacuoles in the posterior half of the body, at the side opposite to that of the macronucleus. Lt. 0·038 mm. Greatest diameter. 0·025 mm.

Found in the rumen of the goat.

This species somewhat resembles *E. mammillatum*, Da Cunha, but differs from it in general shape, in the shape of the posterior processes, and in the number and position of the contractile vacuoles. It also resembles *E. elongatum* and *E. submaimmilatum* in some respects, but differs in many details, which, not being bridged over by intermediate forms are sufficient for the erection of this new species.

9. E. SPINOSUM, sp.n. (Fig. 4).

Body oval, truncate anteriorly and rounded posteriorly. Anterior end narrow and wholly occupied by the peristome. Posterior end with a small inwardly directed spine near the centre. Macronucleus bandlike, tapering posteriorly. Micronucleus not observed. A single contractile vacuole posteriorly. Lt. 0·035 mm. Greatest diam. 0·02 mm.

Found in the rumen of the cow and goat.

10. E. FURCA, Da Cunha.

1914. Da Cunha, *Mem. Instit Oswaldo Cruz*, T. 6, p. 65.

11. E. QUADRISPINOSUM, sp.n. (Fig. 1).

Body elongated, somewhat barrel-shaped, more tapering posteriorly than anteriorly. Anterior end truncate and entirely occupied by the peristome. Posterior end with 4 inwardly curved spines. Macronucleus elongated, bandlike, and tapering posteriorly. A single micronucleus. A single contractile vacuole in the posterior body half near the macronucleus. Lt. 0·035 mm. Greatest diameter. 0·02 mm.

Found in the rumen of the goat.

The present species differs from *E. furca*, Da Cunha, in having four spines, in the position of the contractile vacuoles and in several other minor points.

12. E. DENTATUM, Stein.

1859. Stein, *Lotos*, *Zeitschr. f. Naturwiss.* Prag., Vol. 9, p 57.
1888. Kent, *Manual of Infusoria*, p. 654.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 269.

Schuberg (17) regarded the species as one of *Diplodinium*; but his view cannot be accepted in accordance with Eberlein's researches.

13. E. ROSTRATUM, Fiorentini.

1890. Fiorentini, *Boll. Sci.*, Vol. XII, p. 89.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 270

14. E. CAUDATUM, Stein.

1859. Stein, *Abhandlungen der kgl. Boehmischen Ges. d. Wiss.*, Folge V. Bd. X, pp. 35-38.
1859. Stein, *Lotos*, *Zeitsch. f. Naturwiss.*, Prag., Vol. 9, p. 57.
1888. Schuberg, *Zool. Jahrb. Syst. Abth.*, Vol. 3, p. 409.
1888. Kent, *Manual of Infusoria*, p. 654.
1895. Eberlein, *Zeitschr. wiss, Zool.*, Bd. 59, p. 268-9.
1911. Doflein, *Lehrbuch der Protozoenkunde*, Aufl. 3. p. 977.
1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 65.

15. E. SUBSPHAERICUM, sp.n. (Fig. 10).

Body irregularly and broadly oval, strongly convex dorsally, more or less straight on the other side. Anterior end narrow and entirely occupied by the peristome. Posterior end bent towards the ventral aspect, and provided with fine elongated recurved spines Macronucleus short, stout and rodlike. Micronucleus towards the anterior end. Two contractile vacuoles in the middle of the body. Lt. 0·08 mm. Greatest diameter. 0·05 mm.

Found in the "honey comb" of the cow.

16. E. BICARINATUM, Da Cunha.

1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 65.

17. E, MAMMILO-CARINATUM, sp.n. (Fig. 9).

Body somewhat elongated, wide towards the posterior end. Anterior end narrow and entirely occupied by the peristome, which presents a wide triangular fissure on one side. A slight constriction round the body above the middle of its length. Posterior end with two rounded processes, and a projecting keel on one side. Macronucleus more or less club-shaped. A single micronucleus. A single contractile vacuole (?). Lt. 0·07 mm. Greatest diameter 0·04 mm.

Found in the rumen of the cow.

18. E. CARINO-SPINOSUM, sp.n. (Fig. 5).

Body somewhat barrel-shaped. Anterior end truncate and entirely occupied by the peristome. Posterior end with a keel on one side and with a spine on the other. Macronucleus short and bandlike. A single micronucleus. A single contractile vacuole in the middle near the macronucleus. Lt. 0·035 mm. Greatest diameter 0·025. mm.

Found in the rumen of the goat.

Gen. 2. Diplodinium, Schuberg.

1859. *Entodinium*, Stein, vide supra (pp. ??).
1888. Schuberg, *Zool. Jahrb. Syst. Abt.*, Bd. III, p. 404.
1889. Bütschli, "Protozoa" in Bronn's *Thierreich*, p. 1738.
1896. Delage and Hérouard. *Traité de Zool., concrète*, T. I, p. 468.
1903. Hickson, "Infusoria" in Lankester's *Treatise on Zoology* (Part I, second fascicle), p. 410.

1. D. MINIMUM, Da Cunha.

1895. *D. ecaudatum*, Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 263
1914. Da Cunha, *Mem. Inst. Oswaldo Cruz*, T. 6, p. 61.

2. D. CONICUM, sp n. (Fig. 12).

Body irregularly conical, widest anteriorly and somewhat flattened laterally. Anterior end with a large peristome and a small conical depression for the secondary spiral of membranelles, a large raised lobe separating the two. A second conical process on the longer side of the body displacing the peristome on one side. Posterior end curved and tapering, with a depression in the centre for the anal canal. Ectosarc greatly thickened anteriorly and less so posteriorly round the anal canal. Macronucleus stout, short and bandlike, somewhat tapering below. A single micronucleus at the side of the middle of the macronucleus. Two contractile vacuoles near the macronucleus. Lt 0·11 mm. Greatest diameter 0·075 mm.

Found in the rumen of the goat.

This species comes nearest to *E. minimum*, Da Cunha but differs from it mainly in the presence of an anterior lobe displacing the periostome on one side below the level of the anterior end, in the number of the contractile vacuoles and in the shape of the posterior end

3. D. BURSA, Fiorentini.

1889. Fiorentini, *Boll. Sci.*, Vol. XI, p. 11.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 256-60.
1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 63.

4. D. MAGGII, Fiorentini.

1889. Fiorentini, *Boll. Sci.*, XI, pp. 11-12.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 252.
1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 63.
1914. Sharp, *Berkley Univ. Calif. Publica. Zool.*, Vol. 13, p. 60.

5. D. TUBERCULATUM, sp.n (Fig. 11).

Body elongately oval, wide in the middle tapering anteriorly and rounded posteriorly. Large peristome separated from the conical area of secondary membranelles by a raised lobe. Posterior end with three small unequal lobes, with a slit between them and placed somewhat laterally. Macronucleus elongated, stout and bandlike. Micronucleus near the posterior end of the macronucleus. Two large contractile vacuoles near the macronucleus. Lt. 0·1 mm. Greatest diameter 0·065 mm.

Found in the rumen of the goat.

This species differs mainly from *D. maggii*, Fior., in the presence of two contractile vacuoles, from *D. bursa*, Fior. in the presence of a single micronucleus and from the both in the shape of the macronucleus and in the presence of three posterior lobes.

6. D. CYLINDRICUM, sp.n. (Fig. 13).

Body cylindrical, subtruncate at both ends. Anterior end with three irregular lobes Closely placed small peristome, and the conical area of secondary membranelles displaced beneath the extreme anterior end. Posterior end with a central rounded process surrounded by six short spines directed towards the centre. Macronucleus elongated somewhat dilated anteriorly. A single micronucleus. Two contractile vacuoles on the side opposite to that of the macronucleus. Lt. 0·1 mm. Greatest diameter 0·06 mm.

Found in the "honey comb" of the goat. A single specimen was observed and this was probably in the process of transverse fission.

7. D. ROSTRATUM, Fiorentini.

1889. Fiorentini, *Boll. Sci.*, XI, pp. 11-12.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 262.
1914. Da Cunha, *Mem. Instit Oswaldo Cruz*, T. 6, p. 64.

8. D. EBERLEINI, Da Cunha.

1895. *D. caudatum*, Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 260-1.
1914. Da Cunha, *Mem. Inst. Oswaldo Cruz*, T. 6, p. 62.

9. D. DENTATUM, Fiorentini.

1889. Fiorentini, *Boll Sci.*, XI, pp. 11-12.
1895. *D. mammosum* Raillet, *Traité de Zool. médicale et agricole.*
1895. *D. denticulatum* (young form), Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 261-262.
1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 64.

9 a. D. DENTATUM VAR. FIORENTINII, Awerinzew and Mutafowa.

D. fiorentinii, Awerinzew and Mutafowa. *Arch Protistenk*, Bd. 33, pp. 110-111.

10. D. QUADRIDENTATUM, sp.n (Fig. 14).

Body band-shaped and tapering at both ends. Small peristome on one side of the anterior end. Secondary spiral of membranelles placed at the other end of the anterior end and separated from the peristome by a rounded lobe. Four curved spines at the posterior end and placed at equal distances from one another. Macronucleus elongated bandlike. Micronucleus obscured by numerous oval refractive bodies in the endoplasm. Five (?) scattered contractile vacuoles. Lt. 0·075 mm. Greatest diameter 0·04.

Found in the rumen of the cow.

11. D. DENTICULATUM, Fiorentini.

1889. Fiorentini, *Boll. Sci.*, XI, pp. 11-12.
1895. *D. dentatum*, Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 261-2.
1895. *D. dentatum*, Stein, Railliet, *Traité de Zool. médicale et agricole.*

12. D. ANISACANTHUM, Da Cunha.

1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 64.

GEN. 3. METADINIUM, Awerinzew and Mutafowa.

1914. Awerinzew and Mutafowa, *Arch. Protistenk*, Bd. 33, p. 115.

1. M. MEDIUM, Awerinzew and Mutafowa.

1914. Awerinzew and Mutafowa, vide supra.

2. M. CARINATUM, sp. n. (fig. 15).

Body broadly oval and slightly flattened. Posterior end rounded with a keel on one side. Peristome small and on the side of the anterior end opposite to that of the keel. Secondary spiral of membranelles occupying a small conical area and separated from the peristome by a round lobe and not connected with the latter by a row of membranelles (the main generic character). Macronucleus stout with a dilated end and placed more or less transversely. Endoplasm with numerous oval refractive bodies A single large contractile vacuole in the middle. Lt. 0·15 mm. Greatest diam. 0·092 mm.

Found in the rumen of the cow.

3. M. QUADRATUM, sp. n. (fig. 16)

Body irregularly quadrilateral, somewhat flattened with the ventral side concave and dorsal side convex. Peristome small placed on one side of the anterior end. Secondary spiral of membranelles occupying a very small conical area on the dorsal side of the anterior end and separated from the peristome by a large raised flattened lobe. Posterior end with six recurved spines somewhat arranged in pairs. Ventral pair of spines longer and the dorsal pair shorter than the middle. Macronucleus elongated, stout and rodlike. A single contractile vacuole in the postero-ventral region. Lt. 0·11 mm. Greatest diam. 0·065 mm.

Found in the rumen of the cow.

Gen. 4. OPHIYOSCOLEX, Stein.

1859. Stein, *Lotos, Zeitschr f. Naturwiss*, Prag., Vol. 9, p. 57.
1867. Stein, *Organismus d. Infusionsthiere* Bd. II, p. 164
1888. Kent, *Manual of Infusoria*, p. 653.
1881-9. Bütschli, "Protozoa" in Bronn's *Thierreich*, p. 1738.
1896. Delage and Hérouard. *Tráité de Zool. concrète*, T. 1, p. 468.
1903. Hickson, "Infusoria" in Lankesters *Treatise on Zool.* (Part I, second fascicle), p. 409.

1. O. INERMIS, Stein.

1859. Stein vide, supra.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 240.
1914. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6, p. 61.

Found in the rumen of the goat in Calcutta.

2. O. ECAUDATUM, Fiorentini, sp.

1889. *Diplodinium ecaudatum*, Fiorentini, *Boll. Sci.*, XI, p. 12.
1914. *D. ecaudatum* forma *ecaudatum*, Sharp, *Univ. Calif. Publica. Zool.*, Vol. 13, p. 62.
1914. *O. labiatus*, Awerinzew and Mutafowa, *Arch. Protistenk*, Bd. 33, p. 114.

2*a*. O. ECAUDATUS, var. CAUDATUS, Fiorentini sp.

1889. *Diplodinium caudatum*, Florentini, *Boll. Sci.* (loc. cit.)
1914. *D ecaudatum* forma *caudatum*, Sharp, *Univ. Calif. Publica. Zool.*, Vol. 13, p. 90.
1914. *O. intermixutus*, Awerinzew and Mutafowa, *Arch. Prostistenk*, Bd. 33. p. 112.

Found in the rumen of the cow in Calcutta.

2*a* (*a*). O. ECAUDATUS, var. CAUDATUS. subvar, ELONGATUS (nom. nov.)

1914. *O. inermis* var. *caudatus*. Da Cunha, *Mem. Instit. Oswaldo Cruz*, T. 6. p. 61, fig. 1 (pl. 7).

2*b*. O. ECAUDATUS, var. BICAUDATUS, Sharp.

1914. *Diplodinium ecaudatum*, forma *biscaudatum*. Sharp. *Uni. Calif. Publica. Zool.*, Vol. 13, p. 92.

2*c*. O. ECAUDATUS, var. TRICAUDATUS, Sharp.

1914. *Diplodinum ecaudatum* forma *tricaudatum*, Sharp, *Univ. Calif. Publica. Zool.*, Vol. 13. p. 92.

Found in the rumen of the goat in Calcutta (fig. 17).

2*d*. O. ECAUDATUS var. QUADRICAUDATUS, Sharp.

1914. *Diplodinium ecaudatum* forma *quadricaudatum*. Sharp, *Univ. Calif. Publica. Zool.*, Vol. 13, p. 93.

Found in the rumen of the goat in Calcutta (fig. 18). But the present species differ mainly from the type in the length and arrangement of the spines and in the extent of the transverse girdle of the membranelles.

2*e*. O. ECAUDATUS var. PENTACAUDATUS.

1914. Sharp. *Univ. Calif. Publica, Zool.*. Vol. 13, p. 93.
1914. *O. fasciculatus*. Aworinzew and Mutafowa, *Arch. Protistenk*, Bd., 33, p. 113.

2*e* (*a*). O. ECAUDATUS var. PENTACAUDATUS, subvar. PARVICAUDA, Averinzew and Mutafowa.

1914. *O. fasciculatus* var. *parvicauda*. Awerinzew and Mutafowa, *Arch. Protistenk*, Bd. 33, p. 113.

Found in Calcutta in the rumen of the goat (figs. 19 and 20). It differs from the typical variety (with 5 caudal processes) mainly in the unequal length of the tails and in the extent of the secondary girdle of membranelles.

3. O. BENGALENSIS, sp. n. (fig. 21).

Body elongated, tapering at both ends. Peristome occupying a portion of the anterior end. Secondary girdle of membranelles close to the anterior end and extending half the circumference of the body. Posterior end with two rounded lobes, one projecting beyond the other. Macronucleus short, irregularly pyriform, occupying the middle of the body. A single micronucleus. Two contractile vacuoles, one near the girdle, the other near the posterior lobes. Lt 0·05 mm. Greatest diam. 0·025 mm.

Found in the rumen of the cow.

4. O. SPINOSUS, sp. n. (fig. 22).

Body elongated, barrelshaped. Anterior end wholly occupied by the large peristome. Secondary girdle of membranelles close to the anterior end and extending nearly to half the circumference of the body. Posterior end with a peripheral circular row of six unequal spines. Macronucleus elongated, bandlike. Two large contractile vacuoles on the side opposite to that of the macronucleus. Lt. 0·085 mm. Greatest diam. 0·045 mm.

Found in the rumen of the cow.

5. O. CATTANEOI, Fiorentini.

1889. *Diplodinium cattaneoi*, Fiorentini, *Boll. Sci.*, XII, p. 7.
1914. Da Cunha, *Mem. Inst. Oswaldo Cruz*, T. 6, p. 63.

6. O. CAUDATUS, Eberlein.

1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, pp. 247–250.

Found in Calcutta in the rumen of the goat.

7. O. PURKYNJEI, Stein.

1859. Stein, *Lotos Zeitschr. f. Naturwiss.*, Prag., Vol. 9, p. 57.
1888. Kent, *Manual of Infusoria*, p. 653.
1889. *Diplodinium vortex*, Fiorentini, *Boll. Sci.*, XI, p. 11.
1895. Eberlein, *Zeitschr. wiss. Zool.*, Bd. 59, p. 250.

8. O. EBERLEINI, sp. n. (fig. 23).

Body elongated with a slight necklike constriction anteriorly. Anterior end tapering and nearly wholly occupied by the peristome.

Secondary girdle of membranelles occupying the necklike constriction and very short in extent. Posterior end rather abruptly tapering, with a greatly curved stout spine surrounded by a row of bifid teeth. Macronucleus a stout band. Contractile vacuoles 4 or 5 in number (?). Lt. 0·135 mm. Greatest diam. 0·07 mm.

This species differs mainly from *O. caudatus*, Eberlein, in the single row of bifid teeth and in the extent of the transverse girdle.

Gen. 5. TRIPLODINIUM, gen. n. (figs. 24–25).

Peristome on one side. Two secondary spirals of membranelles on the other side placed in small conical depressions and connected with each other by a row of membranelles.

Found in the rumen of the cow. A single specimen was observed and is named *T. bovis*, sp. n., which may be described as follows:—

Body elongated, slightly tapering anteriorly, more so posteriorly, and somewhat flattened laterally. Peristome small and placed on one side of the truncate anterior end. Posterior end with three somewhat flattened unequal spines. Macronucleus elongated, somewhat fusiform and placed beneath the ectosarc in the anterior bodyhalf. A single micronucleus. Two contractile vacuoles on the side opposite to that of the macronucleus. Lt. 0·05 mm. Greatest diam. 0·03 mm.

II. SUBFAM. TROGLODYTELLINAE, subfam. nov.

No functional peristome. Body with four incomplete girdles of membranelles. No contractile vacuole.

Gen. TROGLODYTELLA, Brumpt and Joyeaux.

1912. Brumpt and Joyeax, *Bull. Soc. Path. Exot.*, Vol. V, pp. 500–503.

1. T. ABRASSARTI, Brumpt and Joyeaux.

1912 Brumpt and Joyeaux, vide supra.

1*a*. T. ABRASSARTI, subspecies, REICHENOW.

1917. Reichenow, *Boll. Soc. Espan. Hist. Nat.*, xvii, pp. 312–4.

2. T. GORILLAE, Reichenow.

1917. Reichenow, vide supra.

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5*a*. Da Cunha, *Mem. Inst. Oswaldo Cruz*, T. 11 (1919), p. 8.
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21. Stein, *Der Organismus der Infusionsthiere*, Bd. II (1867).

THE FIGURES.

The figures are drawn with camera lucida, with 1/6 in. objective (Leitz) and No. 12 compensating eyepiece (Zeiss), except fig. 15, which is drawn with No. 6 Compensating Eyepiece (Zeiss).

EXPLANATION OF PLATE I.

Fig. 1.—*Entodinium quadrispinosum*, sp. n
Fig. 2.—*E. ellipsoideum*, sp. n.
Fig. 3.—*E. ovatum*, sp. n.
Fig. 4.—*E. spinosum*, sp. n
Fig. 5.—*E. carino-spinosum*, sp. n.
Fig. 6.—*E. nudum*, sp. n.
Fig. 7.—*E. submammillatum*, sp. n.
Fig. 8.—*E. elongatum*, sp. n.
Fig. 9.—*E. mammillo-carinatum*, sp. n.

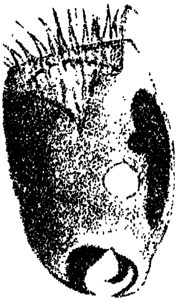

Fig. 1.

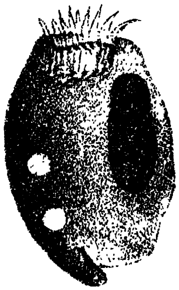

Fig. 2.

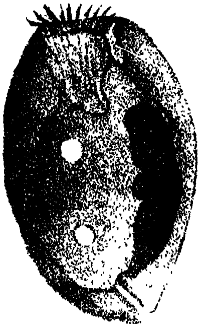

Fig. 3.

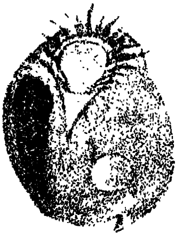

Fig. 4.

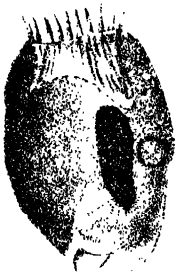

Fig. 5.

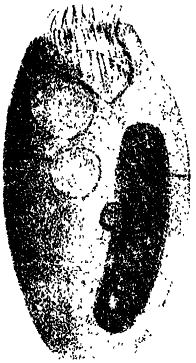

Fig. 6.

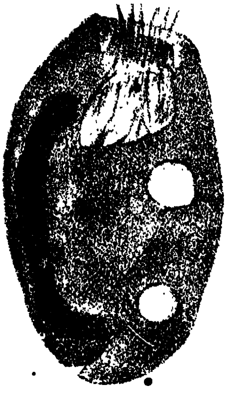

Fig. 7.

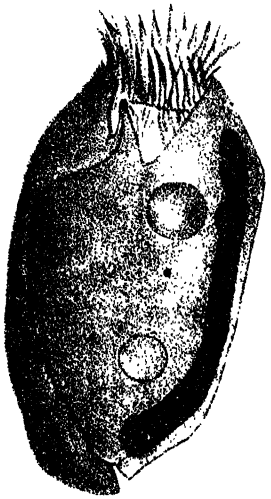

Fig. 8.

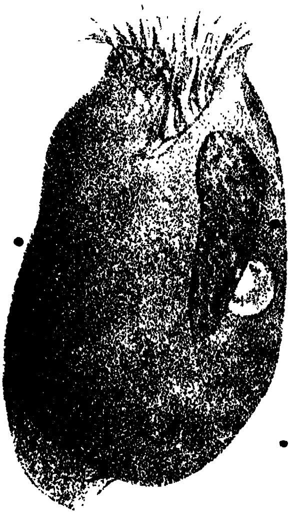

Fig. 9.

E. Ghosh del

EXPLANATION OF PLATE II.

Fig. 10.—*E. subsphaericum*, sp. n.
Fig. 11.—*Diplodinium tuberculatum*, sp. n.
Fig. 12.—*D. conicum*, sp. n.
Fig. 13.—*D. cylindricum*, sp. n.

PLATE II.

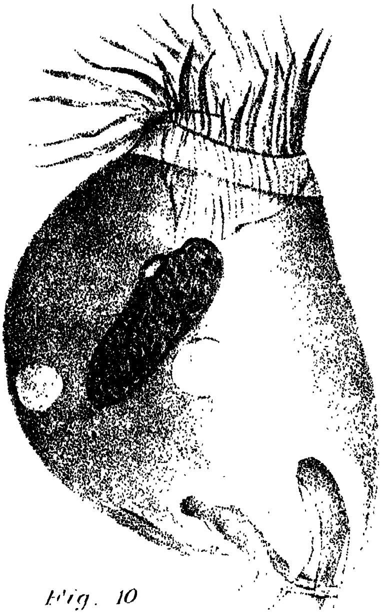

Fig. 10

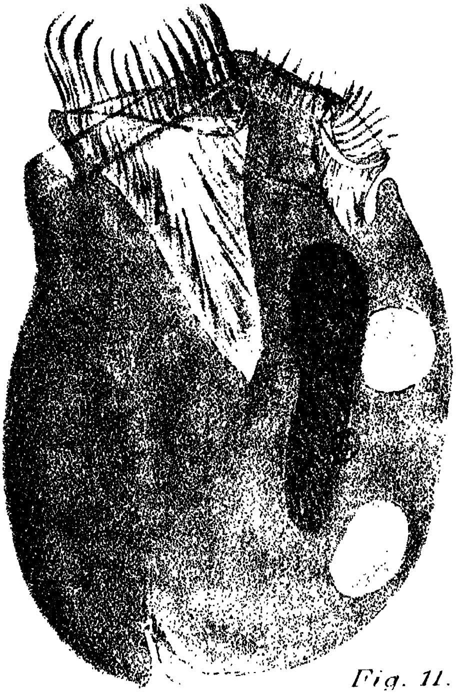

Fig. 11.

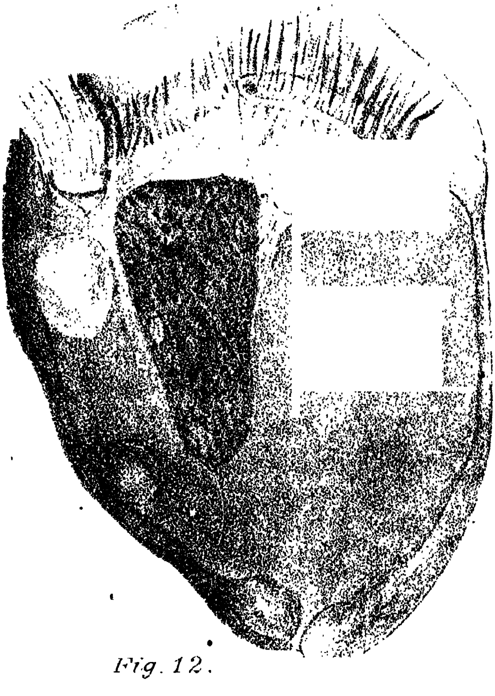

Fig. 12.

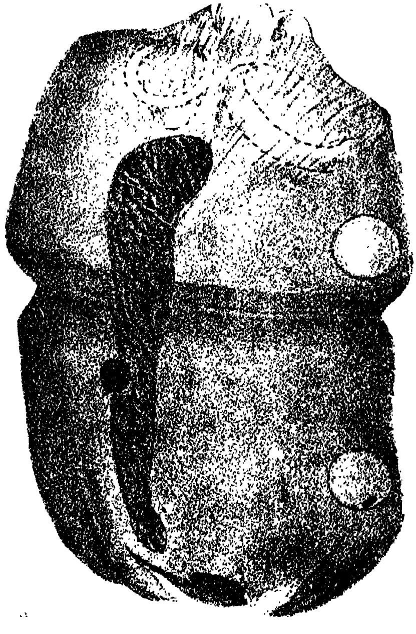

Fig. 13.

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EXPLANATION OF PLATE III.

Fig. 14.—*D. quadridentatum*, sp. n

Fig. 15.—*Metadinium carinatum*, sp. n.

Fig. 16.—*M. quadratum*, sp. *n.*

Fig. 17.—*Ophryoscolex ecaudatus* var. *tricaudatus*, Sharp.

PLATE III.

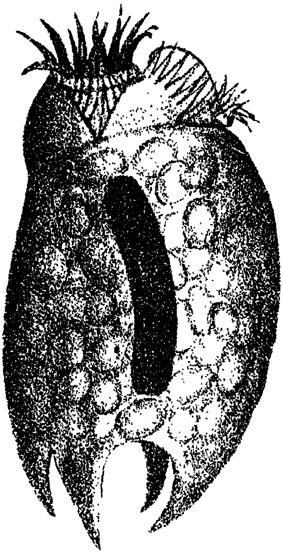

Fig. 14.

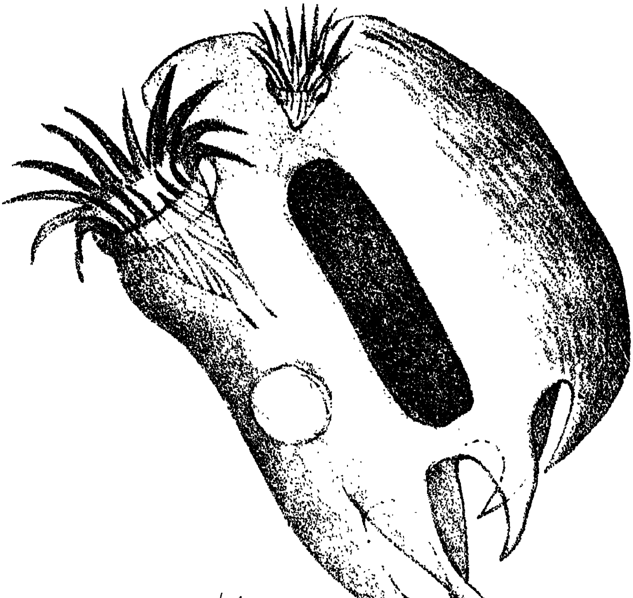

Fig. 16.

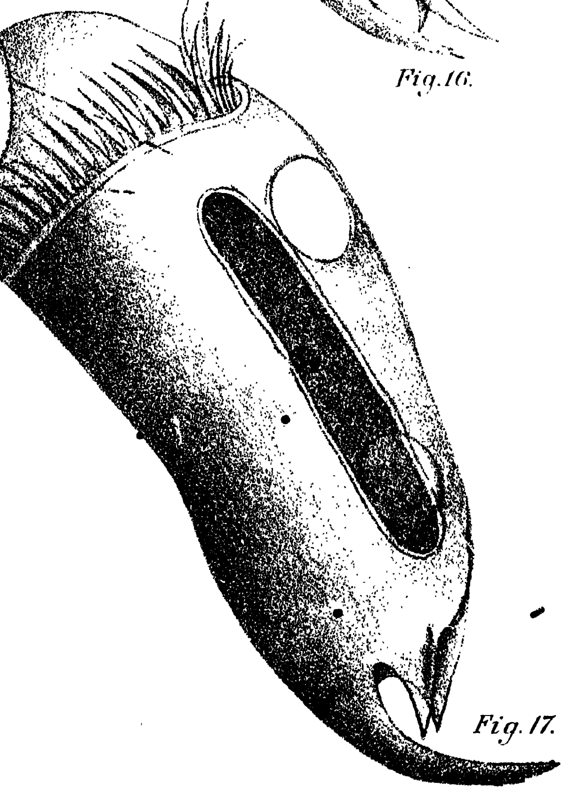

Fig. 17.

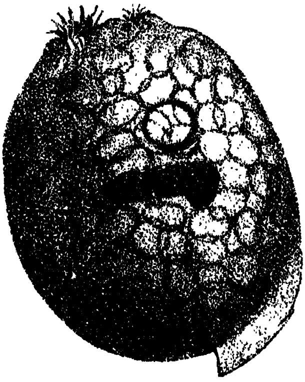

Fig. 15.

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EXPLANATION OF PLATE IV.

Fig. 18.—*O. ecaudatus*, var. *quadricaudatus*. Sharp.

Figs. 19 and 20.—*O. ecaudatus* var. *pentacaudatus*. Sharp sp. Posterior ends.

Fig. 21.—*O. bengalensis*, sp. n.

Fig. 22.—*O. spinosus*, sp. n.

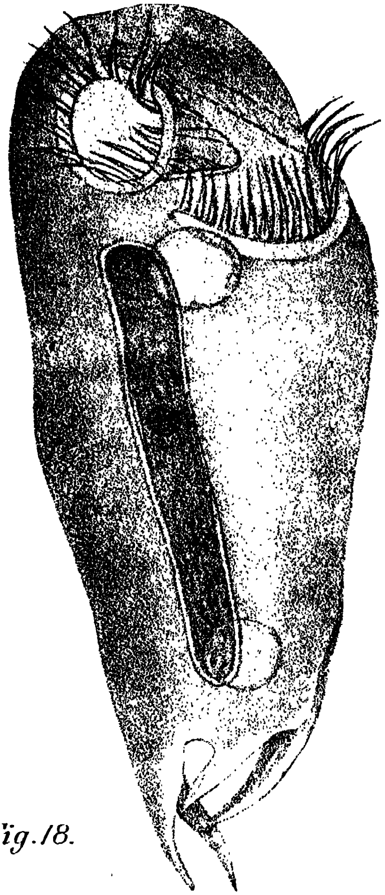

Fig. 18.

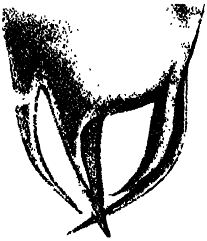

Fig. 19.

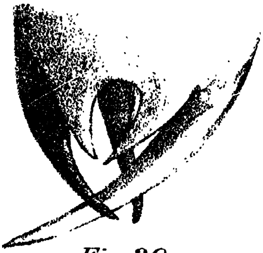

Fig. 20.

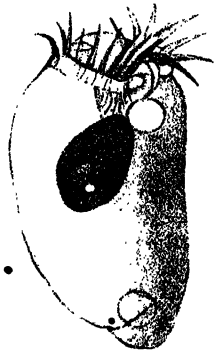

Fig. 21.

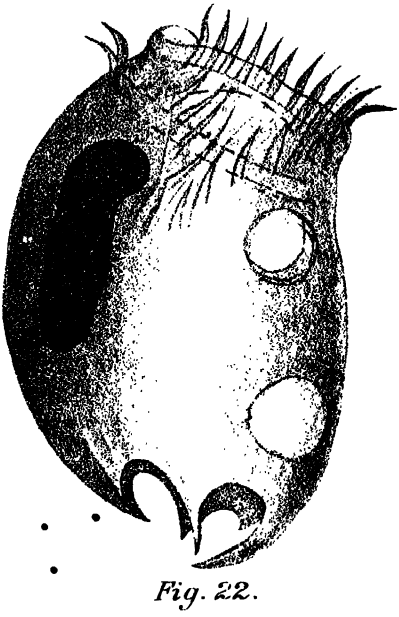

Fig. 22.

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EXPLANATION OF PLATE V.

Fig. 23.—*O. eberleini*, sp. n.

Figs. 25 and 26.—*Triplodinium* (gen. n.) *bovis* (sp. n.)

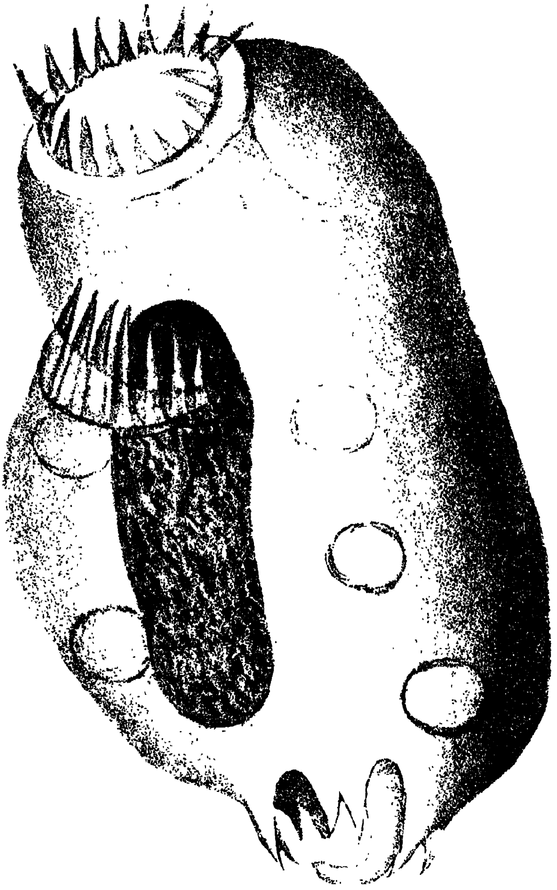

Fig. 23.

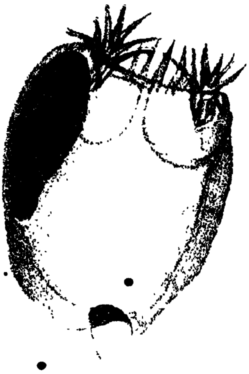

Fig. 24.

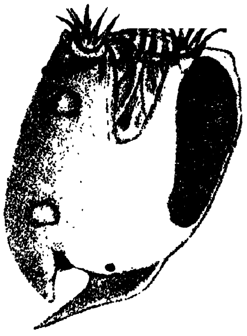

Fig. 25.

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A STUDY IN INHIBITION OF ASSOCIATION.

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Lecturer in Experimental Psychology, University of Calcutta.

The process of inhibition of association has come in for a large share of notice in the psychological forum of to-day. It has long been recognised that all the conditions of association are not given in the series of psychic events. Much that is of importance takes place below the threshold of our ordinary stream of consciousness. The researches of the psycho-analytic school have brought to light many significant and interesting facts that contribute to the inhibition of the associational process. These facts are well known to the student of Psychology and hardly need re-statement.

My task in this paper is not to analyse the ultimate conditions of inhibition in their psycho-physical aspects, nor is it to examine the rôle and significance of the inhibition process itself. My purpose here is to examine the importance of certain processes, given in introspection, that appear in a large number of cases of inhibition. More particularly, I shall try to determine the rôle that the *Aufgabe* plays in inhibition. The experiments upon which I rely here form part of a mere ambitious programme which yet awaits completion.

Materials.—Lists of significant words, each with two associates, one of which was a familiar word and the other a nonsense syllable were made out. The nonsense syllables were "normal." The significant words were two-lettered and the nonsense syllables, all three-lettered. Each series consisted of five original words with their associates. At first however, each series consisted of ten original words. It was found that in some cases, the associates were less liable to reproduction and one of its conditions was the length of the series that occasioned difficulty in learning. The number was reduced to five in order to eliminate this factor. Twenty five-worded series and five series with ten original words were presented—the former to one group of three students and the latter to another group of three. Association cards were prepared, two for every original, with the significant associate on one and the nonsense associate on the other. There was also a series of cards with only the original words.

Procedure.—Each series of association cards was presented to subjects for memorising and the time recorded by stop watch. Reproduction followed every repetition and the number of correct and incorrect reproductions as well as failures were noted. After one set of associations was completely learnt, as tested by complete reproduction twice successively, an associative reaction time was recorded by a stop watch for each original word. Thus there are two R.T's, one for the significant, the other for the nonsense syllable. The significant and the nonsense associates were learnt first in order of time in alternate series. After both of the sets of R.T's had been taken the subject had to look once more through the association cards.

An interval varying from five to eight minutes was allowed as respite at this stage. The subject was then instructed to react by giving a nonsense syllable as each original word was called out. Time was recorded by a stop watch and full introspection was recorded.

Only two five-word series or one ten-word series could be presented on one day. Care was taken to avoid fatigue and distraction as far as possible. All the subjects but one were students of psychology. Three of them had training in experimental work of this nature. The whole series of experiments was preceded by one series of practice experiments, a five-word series, for each subject.

The Problem. Two cases of inhibition might be expected under the conditions of this experiment. In the first place, the desired reaction-word, the nonsense syllable might fail to appear immediately in spite of the instruction given ; in the second place, the meaningful word might appear and the subject, in accordance with his instruction, would have to inhibit it and react by the nonsense syllable. The first of these will be called *involuntary inhibition* and the latter *voluntary inhibition*. It is obvious that voluntary inhibition takes place only when there is an involuntary inhibition. The latter, we might say, is a condition of the former. Our purpose in this paper will be to analyse the conditions of the two kinds of inhibition, subject to the limit that we have put, and to examine the difference, if any, between the two.

Data.—(1) The occurrence of involuntary inhibition could be known either from the actual presence of a rival process, a process other than the desired one, or from the longer reaction time. Taking both of these together there is a wide individual difference in the percentage of inhibitions occurring in the case of the different subjects under the conditions of this experiment.

Percentages to the nearest whole numbers.

Subjects.	A	B	C	D	E	F
Percentages.	70	35	10	40	28	51

FIG. 1.

(2) This difference of percentage cannot adequately be explained in terms of the attentional condition of the different subjects. Those who had a larger percentage of inhibition are not to my knowledge, habitually inattentive. And the daily average of inhibition bear a fair ratio to their respective M.V.'s. Thus, the question of "low" attentional condition of one day offsetting the better attention of another day, is disposed of. In all the four cases where the number of inhibition is large, the M.V. is from 9 to 11 per cent of the daily average. In the case of E the M.V. is larger, about 18 per cent, and there is considerable irregularity of attention.

(3) In most of these cases the fact of inhibition was indicated by the largeness of reaction time as well as by the content of introspection. In some instances, however, the R.T. though large was not abnormally so. The M.V. in such cases gives a better indication of the fact of inhibition than the amount of reaction time.

A few examples of Reaction time and its M.V.

Subjects.	Normal R.T.	Normal M.V.	R.T. in Inhibition.	M.V. in Inhibition.
A	1·6″	·04	2·4″	·6
B	1·2′	·02	4·9″	2
C	1·2″	·1	2·1″	·3
D	1·1″	·04	3·4″	2·2

FIG. 2.

It will be seen from this table that in the case of inhibition, the M.V. is very large. Even when the fact of inhibition is not indicated by the introspective data, it can be inferred from the irregularity of reaction times—if the irregularity be not otherwise accountable. It is interesting to note that this principle has largely been used by the investigators of the psycho-analytic school in their researches in mental pathology.

(4) Introspective analysis of consciousness in involuntary inhibition brings out the following result:

In the first place, the significant associate appears in consciousness as soon as reaction is called for. It has assumed either the verbal form or the visual form. (A detailed account of this will be given below). But along with this dominant image appear certain subsidiary images and sensations. These are generally kinaesthetic, auditory or visual. Thus, there is very often, in the words of the subject B, " a dominant idea with certain marginal processes in which the marginal processes are like overtones that go along with the principal tone in tonal fusion." All the subjects excepting C find this description to be true. No subsidiary process was noticed by C. E, however, does not agree that the subsidiary processes are often in a state of fusion with the dominant image of the significant associate He thinks that it is a process of alternation between the two.

Secondly, not the significant associate as a whole but a partial representation of it or one of the subsidiary processes attending it appears in consciousness. Thus with the subject A it was sometimes a blurred visual image of the first letter of the significant word and sometimes it was an incipient utterance of the word. With the subject D the auditory image of the word appeared several times. These processes " blocked as it were the appearance of the required association.''

Thirdly, the process of inhibition is indicated by a lengthening of reaction-time and this duration between the signal calling for reaction and the reaction is often marked by certain kinaesthetic sensations. These sensations, the subjects sometimes recognise as somehow connected with the meaningful associate. Sometimes, however, though the sensations are present no meaning is attached to them. These sensations are almost always of a kinaesthetic nature arising, as several of the subjects say, from the articulatory mechanism. They are also described as ' sensations of tension.'

A table showing the percentage of occurrence of these types of contents in the case of the different subjects of the experiment is given below :—

Subjects.	Complete image of the meaningful associate.	Partial representation of the associate or subsidiary process.	Delayed reaction time.
A	52	30	18
B	48	27	25
C	4	30	66
D	60	28	12
E	72	16	12
F	39	33	28

Fig. 3.

These figures have been worked out to the nearest whole numbers.

(5) The significant associate appears in consciousness under two principal forms: it appears as a whole or it is partially represented In the former case it appears most frequently as a verbal or as a visual image. There are certain other modes of representation—as meaning of the word, for instance—which have been neglected for the purpose of the present enquiry. These cases were but of rare occurrence; and even then, the subject could detect imagery in the meaning. Hence, I have omitted them from the chart attached. When the significant associate assumed any of the forms indicated above, there were a number of subsidiary processes occurring simultaneously or in close succession. These subsidiary processes were mainly visual, verbal, auditory or kinaesthetic. The meaning process referred to above (referred to in the chart as "other") appeared also as a subsidiary process. The partial representations of the significant associate assumed the same forms as the subsidiary processes. There is however no definite correlation between the images appearing in the complete and in the partial representations of the significant associate. There is a good correlation, ·65, only in the case of the visual images in the two cases. We cannot

attach any significance to this in the absence of any other supporting fact—specially as the number of subjects in this experiment is only six. Even a high degree of correlation, according to Spearman, cannot be proved with less than a dozen of cases.* Thus we can only assert that the processes occurring in the two cases are qualitatively the same.

In the instances in which the fact of inhibition is indicated by a large reaction-time, certain subsidiary processes are often noticed. These however are not always recognised as being connected with the significant associate. Moreover, these processes are predominantly of kinaesthetic character. Verbal, visual and auditory experiences occur only in a very few instances (subsumed under "other experiences" in column III). Much of the kinaesthetic experiences arise from the articulatory mechanism—as evidenced by subjects' introspection. As to the exact nature of the processes, their significance and connection, very little is yielded by introspection. Thus, one calls it an "experience of tension," a second describes it as "an effort to pronounce a word which has not yet assumed any definite shape." These, however, would fit in with any meaning that the kinaesthetic experiences might carry. We are not in a position, then, to determine the precise function of these from the introspective account.

(6) An attempt was made to determine the coefficient of correlation between the images occurring in this series of experiments and those brought out independently by Secor's method and Kraeplin's method. The result does not prove the existence of any correlation in a satisfactory way. The 'R' as determined by Spearman's "foot-rule" formula is about ·32, in the case of visual image, ·23 in the case of verbal image and ·17 in the case of auditory image. The result is inclusive in view of the fact that a low degree of correlation (e.g. R=·20) cannot be demonstrated without about 100 cases at hand."† It is very difficult therefore in regular laboratory studies, where we forced to employ a limited number of trained subjects, to demonstrate a low degree of correlation in a satisfactory manner.

(7) The meaning that I attach to the term "voluntary inhibition" has been indicated above. It is clear that voluntary inhibition can take place either when the significant associate appears as a whole or in a partial manner. The principal factor that enters into this process

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* *Brit. Journal of Psychology*. Vol. II, pp. 89-108.

† *Brit. Journal of Psych.*, op. cit.

Table showing the occurrence of the Significant Associate in the form of different images.

	I APPEARANCE OF THE SIGNIFICANT ASSOCIATE AS A WHOLE % 100.												II PARTIAL REPRESENTATION OF THE SIGNIFICANT ASSOCIATE IN TERMS OF SUBSIDIARY PROCESSES % 100.					III DELAYED REACTION WITH CERTAIN PROCESSES % 100.			
	Visual representation.						Verbal representation.														
Subjects.	Percentage of princ. image.	Percentage of the subsidiary processes.					Percentage of princ. image.	Percentage of the subsidiary processes.										Kin sends from articulatory mechanism.	Kin. sends from other sources.	Other experiences.	Only the long reaction without any clear introspection.
		Vis.	Verb.	Aud.	Kin.	Other.		Vis.	Verb	Aud.	Kin.	Other.	Vis	Verb.	Aud	Kin.	Other.				
A	72	15	42	5	2	8	28	16	6	..	..	6	48	32	12	2	6	47	2	12	39
B	67	12	42	2	4	7	33	17	9	2	1	4	54	22	7	9	8	41	33	..	26
C	..	..	..	..	..	..	100	..	13	..	..	..	..	76	.	.	24	16	..	..	84
D	48	13	23	7	..	5	52	28	..	2	2	7	33	62	..	.	5	..	..	42	58
E	46	19	22	..	..	5	54	27	14	..	..	13	4	43	..	21	32	63	..	..	37
F	25	3	14	..	..	8	75	37	19	8	7	14	..	53	7	13	28	31	..	17	52

FIG. 4.

of inhibition is the 'aufgabe'—the instruction which the subject is asked to carry out when he is asked to react by the nonsense associate. The instruction in this case was given in a double form—positive and negative. The subject was told "Do not react by the meaningful word; react by the nonsense syllable." This double form was adopted partially to avoid the complication introduced by the purely negative instruction; and partly to explicate the real process that was expected to appear in the consciousness of the subject. The first of these ends was but imperfectly achieved in the first series of experiments. The subject perceived the aufgabe either positively or negatively—there was hardly a balancing of the two. Later however, the subject was instructed twice in succession for each series and the positive and the negative forms received attention alternately. Introspection shows that the subject was not acting purely under negative instruction.

(8) The *Aufgabe* operates as an inhibiting factor in several ways In the first place, the aufgabe appears as a "meaning or idea" with which come a number of secondary processes usually verbal, visual and auditory. Thus the subject B says: "There was a general notion that the meaningless syllable was to be given; a verbal utterance, explicit or incipient, of "no" signifying that the significant associate was not to be given, almost always accompanied it." The subject A speaks often of visual phenomena; "grey lines crossing one another appear to move about in the visual field" These phenomena, according to the subject, represent the negative form of the *Aufgabe*.

In the second place, these secondary phenomena, visual, verbal and auditory, together with certain movements make their appearance and inhibit the meaningful associate. The associate disappears in characteristic ways in the case of the different phenomena. Thus, in the case of visual phenomena, the significant associate, represented visually, "moves out of the field of vision." In the case of the phenomena there has often been remarked an alternation of other processes mentioned above and the verbal and auditory representations of the significant associate.

Lastly, the process of inhibition is brought about, through the occurrence of certain movements, actual or incipient, of hand and head. Here, the consciousness of the movements executed or intended, appears to occupy the field of attention for the time being and replaces the significant associate. A table representing the percentage of occurrence of the visual, verbal and auditory phenomena in the process of voluntary inhibition is given below. By the term "mixed" as used here, I mean those instances when the subject could not precisely indicate the nature of the imagery.

Subjects.	Visual	Verbal.	Auditory.	Mixed.
A	67	27	..	6
B	53	44	..	3
C	11	72	9	8
D	37	48	12	3
E	43	46	4	7
F	29	66	..	5

FIG. 5.

(9) A high degree of correlation obtains between the 'principal' visual and verbal image in fig. 4 and those in fig. 5. Although I have spoken about the unsatisfactory character of the R obtained from six subjects, it might be an indication of the qualitative similarity of imagery in the two cases, if of nothing more. The R, as obtained by Spearman's 'foot-rule' formula is ·65 for both the kinds of imagery.

Conclusion.

(1) Inhibition is a process of displacement of one mental content by another. In this case, the meaningful associate displaces the meaningless one from the focus of attention and is in its turn displaced by the aufgabe.

(2) The process of inhibition is often one of alternation between the two mental contents—the inhibiting and the inhibited contents

(3) The inhibiting content is always a complex, constituted of a focal content with certain subsidiary image-processes.

(4) The subsidiary processes may at times take the place of the focal content and serve as the inhibiting factor. Thus, the inhibiting factor appears in varying degrees of complexity—at the one end lies

the focal image of the significant associate or of the aufgabe with its many subsidiary image-processes, at the other end lies a barely noticeable subsidiary image process, with a delayed reaction to indicate the fact of inhibition. Between these lie the inhibiting factors of gradually diminishing conscious complexity.

(5) The voluntary and the involuntary inhibition as defined above are of the same character. In the former instance, the significant associate and in the latter the aufgabe are the inhibiting agencies. The representations of both of these in imaginal form are similar in nature as proved by the sameness of quality of the images in the two cases as well as by the high degree of correlation obtaining between the visual and the verbal images.

To sum up then, the process of inhibition is brought about by a complex of images (under the conditions of this experiment). The complex may function as a whole or may disintegrate and function in part. In the case of partial functioning, the remaining factors are not present in consciousness, are not detected by introspection. Yet their operation cannot be thought to have ceased; for, the reaction time does not decrease with the decrease of the introspective complexity of the inhibiting factor. It can only mean then that the apparently absent constituents are still operating as inhibiting agencies. In all forms of associative inhibition, the process of inhibition simply means that the inhibited content has been displaced by the inhibiting content —that the latter occupies the focus of attention for the time being. Thus, inhibition really is a function of attention. A question may arise as regards the reason of the significant associate and the aufgabe naturally appearing in the focus of attention. The reason to my mind is the complexity of both of these contents. The nonsense syllable yields its place to the meaningful associate because the latter is more complex. The meaningful associate yields to the aufgabe, since the latter is more complex. But mere complexity would not explain: there is another factor namely, the relative liability to reproduction. Liability of Reproduction depends in a large measure upon 'meaning' and the already established associations. Both of these conditions hold good in the cases noted above.

COMPOUNDS OF HEXAMETHYLENETETRAMINE WITH COMPLEX METALLOCYANIDES AND METALLOCYANIC ACIDS.

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AND

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The formation of carbonylferrocyanides by the replacement of one of the six cyanogen groups of potassium ferrocyanide by the carbonyl group was first observed by Mahler and Müller.* Playfair † showed that similar substitution could be effected by a nitroso group, NO, in ferricyanides producing compounds known as nitroprussides. Starting from sodium nitroprusside Hofmann ‡ has obtained a series of compounds in which one of the cyanogen groups of the ferro- and ferricyanides is replaced by NO_2, H_2O, NH_3, SO_3, or AsO_2 giving rise to a series of ironpentacyano derivatives. Manchot and Woringen § have also prepared and described compounds which may be regarded as derivatives of ferrocyanides in which one of the cyanogen groups is replaced by a methylamine, ethylenediamine pyridine, or hydrazine molecule. Similar substitution products with phenylhydrazine, ethylamine, and diethylamine have been described by Biesalski and Hauser.||

Having regard to these facts we tried to replace the nitroso group of sodium nitroprusside with a molecule of hexamethylenetetramine which is a weak monoacidic base like ammonia and resembles the latter in many respects, and has also been found already to combine with various metallic simple and double salts to form compounds similar to those formed by ammonia.¶

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* *Ann. chem. Phys.*, *6*, 17,93 (1889), *C. R.*, *126*, 1421 (1898).

† *Phil. Trans.*, *139*, 477 (1849).

‡ *Zeit. anorg. Chem.*, *10*, 262–276 : *11*, 31–36 ; *11*, 278–287, *12*, 147–168.

§ *Ber.*, *46*, 3514–3521 (1913).

Zeit. anorg. Chem., *74*, 384–388 (1912).

¶ Pratesi, *Gazzetta*, *13*, 437-438 ; Moschatos and Tollens, *Ber* , *24*, 695–696 ; *Ann.*, Though the experiment was conducted under various different conditions, in no case a substitution product belonging to the pentacyano derivatives with hexamine in the complex was obtained. But it was observed that a concentrated neutral solution of sodium nitroprusside, when mixed with a strong solution of hexamethylenetetramine gave a light-brown crystalline precipitate which on analysis was found to be a simple molecular compound of the two. This led us to examine the action of hexamine with other complex ferro- and ferricyanides with the result that a series of double compounds of hexamine with potassium ferricyanide, potassium cobalticyanide and potassium chromicyanide were obtained. Potassium ferrocyanide, on the other hand, did not give any double compound of the kind but when a saturated solution of it was mixed with a similar solution of hexamine a crystalline precipitate was obtained which, on analysis, proved to be simply a partially dehydrated potassium ferrocyanide. Sodium nitroprusside, potassium ferricyanide, and potassium cobalticyanide gave compounds of similar composition, which are expressed by the following formulæ:—

$$Na_2\left|Fe\begin{matrix}(CN)_5\\NO\end{matrix}\right|, (CH_2)_6N_4, xH_2O;$$

$$K_3|Fe(CN)_6|, (CH_2)_6N_4, xH_2O; \quad K_3[Co(CN)_6], (CH_2)_6N_4, xH_2O$$

Potassium chromicyanide, however, gave

$$3K_3|Cr(CN)_6|, 4(CH_2)_6N_4, xH_2O.$$

Potassium ferrocyanide gave

$$K_4|Fe(CN)_6|, 1\tfrac{1}{2}H_2O$$

with no hexamine in the molecule.

Complex metallocyanic acids gave rise to a similar series of additive compounds with hexamine. These are well crystalline and comparatively less soluble than the corresponding compounds of complex

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272, 271–278; Ley *Ann.*, 278, 57-60; Delépine, *C. R.*, *119*, 1211–1213; ibid., *120* 743-745: Schwartz, *Ch. Z.*, *14*, 787; Hofmann, (*Ber.*, *39*, 3181–3184; Bergell, *Ch. Z.*, 1907, i, 487-488: Schmiz, *Ber. d. Ph. G.*, *20*, 201-202, 1910: Calzolari *Ber.*, *43*, 2217-2219; Barbieri and Calzolari, *Atti. R. A.*, 1910, v, *19*, ii, 584–590; ibid., 1911, v, *20*, i, 164-169: 125–129; *Zeit. anorg. Chem.*, *71*, 347-355, 1911; Barbieri and Lanzoni *Atti. R. A.*, 1911, v *20*, i, 161–164; Scagliarini, ibid., 1912, v, *21*, ii, 151–154, 642–643; Calzolari, ibid., 1913, v *22*, i, 787–792; Barbieri, ibid., 1914, v, *23*, ii, 8-12; Rossi, *Gazzetta*, *43*, ii, 669–671, 1913; Scagliarini and Rossi, *Atti R. A.* 1913, v, *22*, ii, 506–508: Calzolari and Tagliavini, ibid., 1915, v, *24*, i, 925-932; Vanino and Schinner, *Arch. d. Ph.* 252, 449–459, 1914; Scagliarini and Marangoni, *Atti. R. A.*, 1914, v, 23, ii, 12-14.

metallocyanides with hexamine. They may be regarded either as simple molecular compounds of the acids with hexamine or as the acid salts of hexamine; which are completely hydrolysed in aqueous solution. No complex ions containing hexamine are formed in aqueous solution; as demonstrated by the measurements of their molecular conductivity.

The following formulæ represent their composition :—

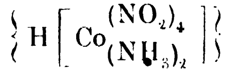
$3H_4Fe(CN_6$, $4(CH_2)_6N_4$, xH_2O : $3H_4Fe(CN)_6$, $5(CH_2)_6N_4$, xH_2O ;
$3H_3Fe(CN)_6$, $4(CH_2)_6N_4$, xH_2O ; $3H_3Co(CN)_6$, $4(CH_2)_6N_4$, xH_2O.

Compounds of hexamethylenetetramine, with ferro- ferri- and cobalticyanic acids of the composition.

$$2C_6H_{12}N_4 \,.\, H_4Fe(CN)_6 \,.\, 7\tfrac{1}{2}H_2O$$
$$2C_6H_{12}N_4 \,.\, H_3Fe(CN)_6 \,.\, 2\tfrac{1}{2}H_2O$$
and
$$2C_6H_{12}N_4 \,.\, H_3Co(CN)_6 \,.\, 3\tfrac{1}{2}H_2O$$

have been prepared by Wagener and Tollens * Evidently the compounds obtained by us are quite different from the above and this has been conclusively proved by the determination of their molecular conductivity in aqueous solution, the results of which are in complete agreement with the molecular formulæ established by us. How far the different results obtained by these workers are due to the actual existence of a different series of compounds as stated by them, and how far they are due to imperfect washing of the crystals or to the impurities present in them as admitted by the workers themselves in their paper cannot be definitely stated.

Interesting results have, however, been obtained in the action of hexamine upon sodium cobaltinitrite, in which it appears to replace one of the NO_2 groups of the complex anion $[Co(NO_2)_6]^{---}$. Compounds with complex acids such as mercuri-iodic acid (H_2HgI_4), chromithiocyanic acid, cadmi-iodic acid (H_2CdI_4) and tetranitro-diammine-cobaltic acid.



$$\left\{ H \left[Co^{(NO_2)_4}_{(NH_3)_2} \right] \right\}$$

have been obtained. A study of these compounds and of the action of hexamine on various other metallic complexes are in progress.

EXPERIMENTAL.

Compound of sodium nitroprusside and hexamethylenetetramine.

Preparation.—A concentrated solution of sodium nitroprusside was mixed with a strong solution of hexamine. Light brown crystals of the double compound at once separated out. These were drained, washed with a little water and dried in the air. They were readily soluble in water giving a brownish red solution.

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* *Ber.*, *39*, 418-419.

Analysis.—The percentage of nitroprusside in the compound was determined by the decomposition of the substance with concentrated sulphuric acid and the estimation of iron. For the determination of hexamine an aqueous solution of a known weight of the substance was treated with a solution of cadmium sulphate. Nitroprusside was precipitated as cadmium nitroprusside. It was filtered off and washed with a dilute solution of cadmium sulphate. The filtrate was then mixed with a known volume of standard sulphuric acid and evaporated to dryness on a water-bath. The residual sulphuric acid was then determined by titration with a standard alkali. From the amount of the acid used up the percentage of hexamethylenetetramine was calculated.*

Sample I.

0·9380 gave 0·1233Fe_2O_3, $Fe=9·18\%$

0·4662 required 3·2729 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4=24·4.$

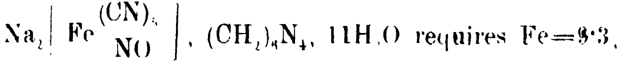
Sample II.

1·1751 gave 0·1528Fe_2O_3, $Fe=9·1\%$

0·4537 required 3·194 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4=24·5\%$



$$Na_2\left|Fe\begin{matrix}(CN)_5\\NO\end{matrix}\right|, (CH_2)_6N_4, 11H_2O \text{ requires } Fe=9·3,$$

and $(CH_2)_6N_4=23·3\%$

Compound of potassium ferricyanide and hexamethylenetetramine.

Preparation.—When a saturated solution of potassium ferricyanide was mixed with a saturated solution of hexamine, beautiful golden yellow crystals of the double compound were obtained. They were then drained, washed with a little water, afterwards with aqueous alcohol and were then dried in the air.

Analysis.—The determination of potassium ferricyanide was carried out by ignition with strong sulphuric acid and estimation of iron.

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* Base. *Ph. Z.*, 52, 851 (1907).

Hexamethylenetetramine was estimated as in the previous compound precipitating off the ferricyanidion as cadmium ferricyanide with a solution of cadmium sulphate.

Sample I.

0·3490 gave 0·0457Fe_2O_3, Fe=9·1%

0·1631 required 1·017 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4$=21·6%

Sample II.

0·4537 gave 0·0586Fe_2O_3, Fe=9 05

0·2815 required 1·7355 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4$=21·43%

$K_3[Fe(CN)_6]$, $(CH_2)_6N_4$, $9H_2O$ requires Fe=8·9 and $(CH_2)_6N_4$ =22·2 per cent.

Compound of potassium cobalticyanide with hexamethylenetetramine.

Preparation.—Saturated solutions of potassium cobalticyanide and hexamine were mixed together. The compound was obtained as a white crystalline precipitate. The crystals were then drained, washed with a little water, afterwards with aqueous alcohol and finally with absolute alcohol. They were then dried in vacuo over sulphuric acid.

Analysis.—Cobalt was estimated, after ignition of the compound with concentrated sulphuric acid, by precipitating it from the solution as cobaltic hydroxide by means of bromine and caustic potash and finally igniting the precipitate in a current of hydrogen and weighing as metallic cobalt. Hexamethylenetetramine was estimated as in the previous case after the removal of the cobalticyanidion as cadmium cobalticyanide with a solution of cadmium sulphate.

Sample I.

0·5960 gave 0·0603 Co, Co=10·1%

0·0962 required 0·637 c.c. $N/1H_2SO_4$,

for conversion into ammonium sulphate,

$(CH_2)_6N_4$=23·0%

Sample II.

0·3631 gave 0·0373Co, Co=10·2%

0·2193 required 1·4632 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4 = 23·2\%$

$K_3[Co(CN)_6]$, $(CH_2)_6N_4$, $6H_2O$ requires Co=10·17 and $(CH_2)_6N_4$ =24·16%

Compound of potassium chromicyanide and hexamethylenetetramine.

Preparation.—Saturated solutions of potassium chromicyanide and hexamine were mixed together. The double compound was obtained as a light yellow crystalline precipitate. The crystals were then drained, washed with a little water, then with alcohol and were dried.

Analysis.—Chromium was estimated as Cr_2O_3 after decomposing the compound with concentrated hydrochloric acid. Hexamine was estimated as before after precipitating off the chromicyanidion as cadmium chromicyanide with a solution of cadmium sulphate.

Sample I.—(air-dried).

0·2266 gave 0·0244Cr_2O_3, Cr=7·4%

0·2095 required 1·78 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4=27·8\%$

$3K_3[Cr(CN)_6]$, $4(CH_2)_6N_4$, $30H_2O$ requires Cr=7·5, $(CH_2)_6N_4$ =27·0%

Sample II.—(vacuum dried).

0·3383 gave 0·0405Cr_2O_3, Cr=8·18%

0·2257 required 1·877 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4=28·99\%$

$3K_3[Cr(CN_6]$, $4(CH_2)_6N_4$, $24H_2O$ requires Cr=7·93, $(CH_2)_6N_4$ =28·9%

Action of hexamethylenetetramine on potassium ferrocyanide.

A saturated solution of potassium ferrocyanide was mixed with a similar solution of hexamine. Yellowish white crystalline precipitate separated out in plates. They were then drained, washed with a little water and dried in the air.

Analysis.—Iron was estimated as Fe_2O_3 after decomposing the compound with concentrated sulphuric acid.

0·5454 gave 0·1118 Fe_2O_3, Fe = 14·29%

0·2716 gave 0·0553 Fe_2O_3, Fe = 14·2%

$K_4[Fe(CN)_6]$, $1\frac{1}{2}H_2O$ requires Fe = 14·18%

Compounds of hexamethylenetetramine with hydroferrocyanic acid.

Preparation.—A solution of freshly prepared pure* hydroferrocyanic acid was made and was mixed with a solution of hexamethylenetetramine. The compound separated out as a milky white crystalline precipitate. The same compound is also obtained on acidifying a mixed solution of potassium ferrocyanide and hexamine. The crystals were rapidly drained, washed with a little water, then with aqueous alcohol and finally with absolute alcohol. They were then dried in vacuo over sulphuric acid. When perfectly dry they are fairly stable but in moist air *the substance* gradually develops a yellow tint and after some days becomes green owing to slow decomposition with the formation of prussian blue.

With an excess of hexamine solution, however, no separation of the above crystals took place. The solution turned slightly yellow and the colour intensified on keeping. On evaporating the solution in vacuo over sulphuric acid, lemon-yellow crystals were obtained. These were drained, washed with a little water, then with aqueous alcohol and finally with absolute alcohol. They were then dried.

Analysis of the white compound.—Ferrocyanic acid was determined by estimating the iron after decomposing the compound with strong sulphuric acid. Hexamine was determined as in the case of compounds of metallocyanides with hexamine, after precipitating out the ferrocyanidion as cadmium ferrocyanide with a solution of cadmium sulphate. The amount of sulphuric acid liberated in this reaction was calculated from the percentage of iron previously determined and this was taken into consideration in calculating the percentage of hexamethylenetetramine.

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* A solution of potassium ferrocyanide was acidified with concentrated hydrochloric acid, the precipitated potassium chloride was dissolved by the addition of more water. Ferrocyanic acid was then precipitated by means of ether. The crystals were then drained and washed with hydrochloric acid and ether. They were afterwards dissolved in absolute alcohol, filtered and reprecipitated with ether.

Sample I.

0·3544 gave 0·0597 Fe_2O_3, Fe=11·79%

0·1846 required 1·987 c.c. $N/1H_2SO_4$, $(CH_2)_6N_4 = 37·5$.

Sample II.

0·3036 gave 0·0507 Fe_2O_3 Fe=11·7%

0·3435 required 3·5607 c.c. $N/1H_2SO_4$,

for conversion into ammonium sulphate,

$(CH_2)_6N_4 = 37·2\%$

$3H_4[Fe(CN)_6]$, $4(CH_2)_6N_4$ $13H_2O$ requires Fe=11·66 and $(CH_2)_6N_4$ =38·8%

Analysis of the yellow compound —The same procedure was adopted as in the previous case.

Sample I.—(air-dried).

0·2967 gave 0·0461 Fe_2O_3, Fe=10·87%

0·1333 required 1·778 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4 = 64·4\%$

$3H_4[Fe(CN)_6]$, $5(CH_2)_6N_4$, $10H_2O$ requires Fe=10·86, and $(CH_2)_6N_4$ =45·8.

Sample II. (vacuum-dried).

0·5934 gave 0·0993 Fe_2O_3, Fe=11·7%,

0·2067 required 2·936 c.c. $N/1H_2SO_4$

for conversion into ammonium sulphate,

$(CH_2)_6N_4 = 49·4\%$

$3H_4[Fe(CN)_6]$, $5(CH_2)_6N_4$, $4\frac{1}{2}H_2O$ requires Fe=11·75 and $(CH_2)_6N_4$ =49·0%

Compound of hydroferricyanic acid and hexamethylenetetramine

A solution of potassium ferricyanide was mixed with one of hexamine. No precipitation of the metallocyanide compound was possible at the dilution employed. On acidification with dilute hydrochloric acid bright yellow crystals separated out from the solution. The crystals were drained, washed with a little water, afterwards with alcohol and were then dried in vacuo over sulphuric acid. The substance is quite stable in dry air, but in the moist air slowly decomposes after a long time.

Analysis.—Iron was estimated after decomposition with concentrated sulphuric acid. Hexamine was estimated in the same way as in the case of the ferrocyanic acid compound after precipitating off the ferrocyanidion as cadmium ferricyanide with a solution of cadmium sulphate.

$$\left\{\begin{array}{l}0\cdot3542 \text{ gave } 0\cdot0565\,Fe_2O_3,\ Fe=11\cdot20\% \\ 0\cdot2883 \text{ gave } 0\cdot0461\,Fe_2O_3,\ Fe=11\cdot22\%\end{array}\right.$$

$$0\cdot1654 \text{ required } 1\cdot7384 \text{ c.c. } N/1H_2SO_4$$

for conversion into ammonium sulphate,

$$(CH_2)_6N_4=36\cdot56\%$$

$3H_3[Fe(CN)_6]$, $4(CH_2)_6N_4$, $16H_2O$ requires $Fe=11\cdot25$ and $(CH_2)_6N_4$ $=37\cdot5\%$

Compound of cobalticyanic acid with hexamethylenetetramine.

Preparation.—A solution of potassium cobalticyanide was mixed with a solution of hexamethylenetetramine, no precipitation taking place at the dilution employed. On acidifying the solution with dilute hydrochloric acid white crystalline precipitate was obtained. It was then drained, washed with water, then with alcohol and were dried in vacuo over sulphuric acid.

Analysis.—Cobalt was estimated as in the case of the potassium cobalticyanide compound. Hexamine was estimated in the same way as in the previous metallocyanic acid compounds, after precipitating out the cobalticyanidion as cadmium cobalticyanide with a solution of cadmium sulphate.

$$0\cdot2986 \text{ gave } 0.0375 \text{ Co},\ Co=12\cdot6\%$$

$$0\cdot1546 \text{ required } 1\cdot8815 \text{ c.c. } N/1H_2SO_4,\ (CH_2)_6N_4=42\cdot3\%$$

$3H_3[Co(CN)_6]$, $4(CH_2)_6N_4$, $9H_2O$ requires $Co=12\cdot86$ and $(CH_2)_6N_4$ $=40\cdot7\%$

Conductivity Measurements.—Electrical conductivity of a solution of hexamine in conductivity water as well as of solutions of some of the above compounds were measured. The results are given below with the molecular conductivities of some of the metallocyanides and metallocyanic acids for comparison.

Cell-constant=0·167.

	Resistance in Ohms.	Concentration in grams per 100 c.c.	Molecular dilution in litres.	μv in reciprocal ohms.	Temp. in centigrade.
$(CH_2)_6N_4$—	Over 50,000	0·0388	....	Negligible	25·80
$K_3Co(CN)_6, (CH_2)_6N_4, 6H_2O$	450	0·0590	999	370·8	25·35
$3H_3Co(CN)_6, 4(CH_2)_6N_4, 9H_2O$	190	0·1312	1049	922	25·35
$Na_2Fe(CN)_5No, (CH_2)_6N_4, 11H_2O$	980	0·0464	1385	236	25·80
$K_3Fe(CN)_6, (CH_2)_6N_4, 9H_2O$	235	0·1062	592	421	25·65
Do	430	0·0531	1184	460·2	26·30
$3H_4Fe(CN)_6, 5(CH_2)_6N_4, 4\frac{1}{2}H_2O$	210	0·1036	1380	1098	25·70
$3H_4Fe(CN)_6, 4(CH_2)_6N_4, 13H_2O$	40	0·7511	192	801·6	25·00
Do.	65·5	....	384	963·2	25·00
Do.	112·5	....	768	1140·0	25·00
Do.	195	....	1536	1314·5	25·00
$3H_3Fe(CN)_6, 4(CH_2)_6N_4, 16H_2O$	45	0·7409	201·4	747·4	25·00
Do.	76	....	402·8	887·7	25·00
Do.	127·5	....	805·6	1055·1	25·00
Do.	215	....	1611·2	1251·5	25·00

Molecular conductivity of Metallocyanides and Metallocyanic acids at 25°c		Molecular dilution in litres	Observed by Burrows and Turner.*
$Na_2Fe(CN)_5.NO$	236·2	1024	
$K_3Fe(CN)_6$	490	1024	Walden.†
$H_4Fe(CN)_6$	1214	512	

As a molecule of the ferro-, ferri- or cobalticyanic acid compound contains three acid molecules, the dilution for a molecule of these compounds corresponds to one-third of that for a molecule of the acid. For example, the dilution of 1380 litres for a molecule of the yellow ferrocyanic acid compound corresponds to $\frac{1380}{3}$, i.e. 460 for the ferrocyanic acid. The value thus obtained, fairly agrees with that for the free hydroferrocyanic acid at the corresponding dilution.

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* *T. Chem. Soc.*, 1919, *115* and *116*, 1431.

† From Kohlrausch's Tables.

The results of these measurements confirm the additive nature of the compounds obtained and establish their molecular formulae as arrived at from chemical analysis.

THE COAGULATION OF METAL SULPHIDE HYDROSOLS.

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The conditions determining the formation and precipitation of colloidal solutions are of fundamental importance in colloid-chemistry. Of the class of colloids generally known as suspensoids or irreversible colloids, arsenious sulphide hydrosol has been investigated more thoroughly than others, and some of the main generalisations regarding this class of colloids rest mainly on experiments made with it.

It is now generally held that the stability * of these sols, that is, their continued existence in a state of fine subdivision is intimately connected with the electric charge of the particles. Some refer it to a kind of protective action which the electric double layer (supposed to exist at the interface) exerts on the particles. Regarding the origin of the charge we have no clear idea though different suggestions have been made. The different views may be roughly indicated as follows :—

(1) The charge is due to a sort of contact electrification—the colloidal particles carrying charge of one sign and the adjacent liquid layer having an equal and opposite charge.

(2) The colloidal particles are "macroscopic" ions resulting from the dissociation of complexes with ordinary ions of opposite charge.

(3) The charge is due to a selective adsorption of ions existing in the liquid.

The only theory that attempts to give a connected account of the process of coagulation is that due to Freundlich. The 'adsorption theory' as this is called rests on the close connection between adsorbability of an ion and its coagulating power showed by Freundlich and his collaborators. The theory assumes that the same rate of coagulation is due to adsorption of equivalent amounts of the oppositely charged ion. The adsorption theory does not attempt to explain the mechanism of coagulation and it has been put forward that coagulation is due to an asymmetry in the charge of different particles* Smoluchowsky holds that this is rather improbable. He has formulated the process of coalescence of the particles in terms of their Brownian movement or diffusion.† He does not attempt to connect theoretically the electrolyte concentration with the rate of coagulation.

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* These sols are not ordinarily stable in the thermodynamic sense.

It was pointed out by Hardy that the colloid separates totally from the sol at the iso-electric point, that is, when the colloid particles carry no electric charge Burton in his book *Physical Properties of Colloidal Solution* mentions experiments supporting this. Work on the migration velocity of colloidal particles in an electric field has shown that even when coagulation takes place immediately after the addition of an electrolyte the particles carry electric charge. It would appear from the sequel that instantaneous coagulation can take place before the iso-electric point has been reached,‡ regards the forces that cause coalescence to be similar to those which give rise to surface tension and holds from the analogy with the well-known case of mercury that with decrease in electric charge these forces should increase in intensity and the maximum is reached at the iso-electric point. One may conclude from this that substances which diminish the surface tension would help coagulation. Kruyt and Duin § carried an experiment with substances like alcohol, phenol and others and arsenious sulphide sols but did not find any definite evidence of such an effect. It is necessary to point out that these so-called *Kapillaractive* substances diminish the tension of liquid-air surface and no conclusion can be drawn as to what influence they may have on the colloidliquid interface.

It would appear from the sequel that the quality of the hydrosol is of great interest and that sufficient attention has not been paid to this.

Methods of Measuring the Coagulating Power of Electrolytes.

It has been pointed out‖ that the order of the coagulating powers of electrolytes given by Linder and Picton and by Freundlich for arsenious sulphide sol does not agree. The methods used by them for measuring the coagulative power are defective. A simple and sensitive method consists of noting the times at which definite changes in the sol take place. This procedure is also free from the defects of previous methods. It is interesting to note that results obtained in this way have been confirmed by later work * using other methods.

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* *Freundlich trans.*, Faraday Soc., 9 (1913), 65.

† *Zeit. phys. Chem.*, 92 (1917), 129.

‡ Bredig, *Anorganische Formente* (1901).

§ *Koll. Chem. Beihefte* 5, (1914), 269.

‖ *J. Amer. Chem. Soc.* 37 (1915), 2024.

Conditions affecting the rate of Coagulation.

(*a*) The distance between the particles of the sol.

Linder and Picton and Freundlich showed that the greater the sulphide content of a sol the greater the quantity of electrolyte necessary for coagulation. This has been adduced in support of the adsorption theory.

It is assumed that the greater the amount of sulphide in a given volume the greater is the sulphide-liquid interface. An increase in surface requires that a greater amount of the coagulating ions is necessary to produce the same surface concentration. Relatively high concentrations of uni-univalent electrolytes are necessary for coagulation The amount adsorbed being small, the concentration of electrolyte necessary for coagulation would be independent of the sulphide content of the sol. This is generally stated to be the case. For electrolytes with multi-valent cations, very dilute solutions suffice for coagulation. The amount adsorbed is comparable to the total quantity of electrolyte present in these cases. In agreement with this it has been observed by Freundlich and others that the amount of electrolyte necessary for coagulation is roughly proportional to the colloid content.

It has been shown however† that these statements are not justified by facts and that the distance between the particles have an equally important effect. These two factors have opposite effects. With copper sulphide and mercuric sulphide hydrosols high concentrations of coagulants are necessary for instantaneous changes in the sol. The amount of electrolyte adsorbed is negligible in comparison to the total amount present. In these cases changes in surface have no perceptible effect and the distance between the particles have a marked effect. With arsenious sulphide sol under suitable conditions the distance between the particles has been shown to exert a greater influence than the change in surface even with an electrolyte like $BaCl_2$ which requires a low concentration of electrolyte for coagulation.

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* Young and Neal, *J. Phys. Chem.*, 1917, 1-13, Kruyt and Spik, *Koll. Zeit.*, 1919 (July), 1-20.

† Mukherjee and Sen, *T*, *115* (1919).

The fact that an increase in distance between the particles increases the stability of the sol leads to several interesting conclusions. Woudotra * pointed out that the Brownian movement of the particles being the cause of collisions between the particles, a change in distance would mean a change in the frequency of the collisions of the particles. An increase in distance between the particles will reduce the number of collisions in unit time and thus reduce the facilities for coalescence. This can be calculated from Smoluchowski's equation.†

Further for arsenious sulphide sols there is an electrolyte concentration which produces instantaneous coagulation, that is, the separation of the colloid in visible big flakes. This undoubtedly shows that the collisions under ordinary conditions are very large in number. Consequently the time taken for the formation of big flakes from a large number of particles is practically *nil*. On increasing the distance between the particles (say by dilution) instantaneous coagulation still takes place though the minimum electrolyte concentration is higher in this case. The possible explanation is as follows :—

The minimum concentration of an electrolyte necessary for immediate coagulation will be called the coagulating concentration. If N_1 be the number of collisions between two particles of a sol at any moment we have that N_1 is a very large number. The truth of this will be apparent if one looks at the particles under an ultramicroscope. Let us assume that on diluting a sol, the individual particles do not change in any respect except that the distance between the particles increases. The number of collisions will change from N_1 to N_2 though N_2 is still a very large number as instantaneous coagulation is possible. Under ordinary conditions the majority of these collisions do not lead to a coalescence of the particles. With increasing concentration of an electrolyte, the percentage of collisions, resulting in coalescence of the colliding particles, increases rapidly till they become so numerous that the coagulation is instantaneous. Let us denote this lowest value of successful collisions that produce the observed effect of instantaneous coagulation by N. The percentage of successful collisions is determined by the potential of the double layer at the surface of the particle; that is, by the electrolyte concentration. Considering the same set of particles and the same electrolyte the higher the electrolyte concentration the greater the percentage. If C_1 and C_2 be the coagulating concentrations (as defined above) corresponding to the sols having N_1 and N_2 number of collisions at any moment then we have that both have the same number of successful collisions, that is, n. And if P_1 and P_2 be the percentage of successful collisions corresponding to the electrolyte concentrations C_1 and C_2 (in the bulk of the liquid), then $P_1N_1 = P_2N_2 = n$.

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* *Zeit. phys. Chem.*, *61* (1908), 607.

† Loc. cit.

Since $N_2 < N_1$ we can see that $C_2 > C_1$. In short the explanation is that the smaller number of collisions on increase in the distance between the particles is balanced by an increase in percentage of successful collisions due to an increase in the electrolyte concentration.

Evidently the neutralising effect of an increased concentration of electrolyte is inexplicable if coagulation takes place at the iso-electric point. From the adsorption theory alone also this cannot be explained as mentioned before. These have been confirmed by Kruyt and Spek* and they have put forward a similar explanation. They do not draw the important conclusion that instantaneous co-agulation takes place before the iso-electric point is reached. Moreover, it is apparent that there is no "definite critical" value for the potential of the double layer above which the sol is stable and below which it is unstable but that the coagulation time is simply governed by the total number of successful collisions.

(*b*) The anomalous role of dissolved hydrogen sulphide.

It is well known that the sulphide sols are generally prepared with hydrogen sulphide and sols rich in sulphide coagulate in its absence. In view of these facts it is difficult to explain the complex effect that dissolved hydrogen sulphide has on the rate of coagulation of these sols with electrolytes. The electrolyte and the sulphide both play an important part in determining the influence of hydrogen sulphide. In the following table the results are given. In some cases the presence of hydrogen sulphide requires a higher concentration of electrolyte for coagulation, i.e. makes it more stable. In other cases the reverse is the case. It is needless to add that the same thing happens if the times for coagulation are noted.

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* Loc. cit.

TABLE 1.

Sulphide.	Electrolytes which show greater stability when hydrogen sulphide is present.	Electrolytes which show less stability when hydrogen sulphide is present.
1. Arsenious Sulphide ..	KCl, HCl, NH_4Cl, LiCl, $Al_2(SO_4)_3$.	$BaCl_2$, $CaCl_2$, $SrCl_2$, $MgSO_4$
2. Mercuric Sulphide ..	KCl, $Al_2(SO_4)_3$	$BaCl_2$.
3. Cupric Sulphide ..	$BaCl_2$	KCl, $Al_2(SO_4)_3$

This anomalous effect of hydrogen sulphide is without parallel in so far as it is the substance which is used in the preparation of all these sols.

On the other hand, experiments with slight traces of alkali and some other substances of a complex nature show the same protective effect for the electrolytes examined.

A detailed discussion of these complex observations is not possible unless certain co-related facts are known. These are the changes dissolved hydrogen sulphide bring about in the charge of the particles, their size, number and hence the total surface of the colloidal particles. But certain conclusions can be drawn from the facts already known. The possible influences of dissolved hydrogen sulphide may be divided into two classes :—

(1) The nature of the colloid is changed. This has a common influence on the process of coagulation independent of the nature of the electrolyte.

(2) The coagulative power of the ion is changed.

The different behaviour of the coagulating ions can then be explained by assuming that their adsorbability changes in the presence of hydrogen sulphide. Adsorption being selective in nature it is quite possible that the change in adsorbability is different for the various salts. To explain the opposite effects shown by the same electrolyte with two different sulphides it will be necessary to assume that the change in adsorption is different for different sulphides. That is, if in presence of hydrogen sulphide the adsorbability of potassium chloride by copper sulphide increases, it is quite possible that it decreases in the case of arsenious sulphide. Such explanations show simply that nothing definite is known of the nature of the adsorption process and anything may be expected. They also show that the so-called adsorption theory fails to give a definite idea of the process of coagulation.

(*c*) Temperature.

The influence that changes of temperature may have on the rate of coagulation is of importance for a correct understanding of its mechanism. Unfortunately nothing has been done on the subject. In a recent communication to the Chemical Society (London) it has been shown that temperature has an anomalous effect similar to that observed with hydrogen sulphide, but the effect is more complicated. The electrolytes can be divided into three classes :—

(1) A higher concentration of the electrolyte is necessary for the same rate of coagulation at a higher temperature. Potassium chloride, potassium sulphate, potassium nitrate, lithium chloride, sodium chloride all show this effect.

(2) At the higher temperature a lower concentration of the electrolyte is necessary for the same rate of coagulation. Barium chloride, calcium chloride, strontium chloride and magnesium sulphate belong to this group, as also hydrochloric acid and sulphuric acid.

(3) The observed effect is complex and varies with electrolyte concentration, quality of the sol and the range of temperature change. Aluminium sulphate and thorium nitrate belong to this class. Sometimes a higher concentration is necessary sometimes a lower.

Experiments showed that hydrogen sulphide produced by increased decomposition of the sulphide at higher temperature cannot explain the facts. A change in adsorbability cannot also explain the different effects observed with different concentrations of the electrolyte or with a variation in the quality of the sol.

These complicated results show the inadequacy of the adsorption theory to explain them. Only a thorough investigation of the process of coagulation in all its aspects can lead to any satisfactory explanation of its mechanism.

ON THE DISINTEGRATIVE FUNCTION OF ATTENTION.

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Attention, according to Wundt, is a process essential to apperception. Our consciousness of meaning in its most complex formation, is apperceptive at its basis. It follows then that attention to any fact would lead to the enrichment of its significance. This is the general law governing the relation between meaning and attention; and it is independent of the validity of Wundt's assumptions concerning the attentional process, inasmuch as it is verified by a large mass of facts even in our every day life.

The very certainty of this general law makes it imperative for us to study any fact that stands out as exception. The purpose of this paper is to study an exception of this sort.

Many of us have noticed that a prolonged fixation upon a meaningful word renders it meaningless and transforms it into a group of meaningless letters. Sometimes even the letters grow strange and unfamiliar. Such phenomena occur in the field of visual attention, no less in the case of auditory attention. It is our purpose in this paper to present a preliminary study of this fact.

Though a matter of common occurrence, the phenomenon has received but scant notice. The only study which we have come across is by Washburn and Severance in the *American Journal of Psychology* 1907.* The materials with which Washburn carried on the study were six-lettered words, in long primer type cut out of the same periodical. These were placed on a white back ground. There was no capital letter. There were six subjects all trained in Experimental Psychology. Each subject had to fixate the word presented for three minutes at the end of which introspection of the changes was taken. The time of 'loss of meaning' was recorded by a stop watch. The average time, as calculated from the data given in the article referred to, for three cases is 22·3 sec. There are four significant changes noticed by Washburn: (1) after a few seconds' fixation, the letters constituting the words arrange themselves into two or more groups; sometimes, a letter or two pass out of notice; (2) some of the letters undergo spatial distortion; some of them, such as *p*, *d*, *b*, etc., turn upside down, while others such as *k*, *l*, *t*, assume a slanting posture; (3) attention is focussed on the artificial groups of letters mentioned above; (4) foreign associations crop up and temporary meanings are attached to the groups of letters thus rendering the total word meaningless.

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* Pp. 182–186.

The following conclusions are drawn by Washburn from the data:

(1) The loss of meaning sets in with a visual re-arrangement of the letters. With the visual re-arrangement come its auditory—motor correspondents. If the normal visual arrangement corresponds to the auditory-motor factor, there is a prolonged meaningful state.

(2) Some of the letters and syllables turn out prominently—they give a turn to the words. Thus *a*, *s*, *m* become prominent.

(3) When a word is unphonetically spelled it suggests the sound of letters rather than of the total word.

(4) Meaning is dependent upon the continuance of the total sound image of the word.

(5) The sound image disappears with the shifting of visual attention.

The experiment described above is capable of improvement in many directions. In the first place, if the visual factor plays an important rôle in the causation of the phenomena, it is but natural that the spatial interval between the constituents of a word would be of great significance. Secondly, the number of letters would surely affect the process of loss of meaning. Thus, the stimuli should be prepared so as to form a graduated series with respect to these factors. Again, the time required for bringing about the state of meaninglessness would indicate how far each of these factors is important for the phenomenon. If the time needed be the same for three-lettered words as for the four-lettered ones, it would mean that the number of letters is not a determinant of the process. If on the other hand, we find that corresponding to a graduated series of words (with respect to the number of letters) there is a graduated series of times, the number of letters must be regarded as being of importance. The same thing holds true of spatial interval between the letters of a word. Lastly, if the phonetically and non-phonetically spelled words, consisting of the same number of letters do not differ as to the time required for bringing about a state of meaninglessness, it would mean that the sound image of the words does not play an important rôle. The following study is calculated to throw light on some of these problems.

Materials.—(1) Two series of words for exposure were prepared. Each series consisted of five sub-series of three, four, five, six and eight-lettered words. Each sub-series had five words for exposure. Two words in each sub-series were phonetically spelled. Jastrow's Memory apparatus was used for exposure and time was recorded by means of a stop-watch. There were four subjects all trained in experimental work.

(2) Four series of words for exposure were prepared. There were, as before, three, four, five, six and eight-lettered words—five of each kind in a series. There were also twelve phonetically spelled words in each series. Each series was divided into five sub-series, but the constituents of a sub-series were not words with the same number of letters. There were three, four, five, six and eight-lettered words in each sub-series. Their order of presentation was irregular. Jastrow's 'memory apparatus' was used, as before, for exposure. The time was recorded by stop-watch. There were three subjects who had studied Psychology, but were not trained in experimental work.

(3) Two series of six-lettered words were prepared for studying the effect of the space interval between the letters. Normally there was a space-interval of 2 mm. between the letters. In one of the series, this normal interval was used; but in the second an interval of 4 mm. was given. The same words were used in the two series. The order of presentation was irregular. The words used were written in ink in capital letters throughout the investigation. Only three subjects, *E*, *F*, *G*, worked for this part of the investigation. The number of words in each series was 10.

Procedure.—The subject sat before the exposure apparatus with his eyes closed. A stop watch was placed in his hand to be started as soon as he saw the word exposed, and to be stopped as soon as the loss of meaning set in. Another stop watch was used by the experimenter for recording the total time of exposure. This procedure was followed throughout the investigation. Introspection was recorded after the exposure-aperture of the Memory apparatus had been closed. The subject was not allowed to see the time recorded by him. An irregular interval, in no case less than three minutes, was allowed as respite between exposures. A few practice exposures were given on each day. Not more than two sub-series was given on one day. It was a "procedure with knowledge."

Data.—(1) The figures given below indicate the mean time (in seconds) to the nearest whole numbers required, in the case of the 7 subjects of the experiment, for the setting in of the loss of meaning.

Subjects.	Av. for 3-lettered words.	M.V.	Av. for 4-lettered words.	M.V.	Av. for 5-lettered words.	M.V.	Av. for 6-lettered words.	M.V.	Av. for 8-lettered words.	M.V.
A	106	21	69	7	64	7	40	6	40	8
B	37	1	23	6	18	3	16	2	9	1
C	53	6	39	8	38	4	33	5	21	2
D	28	5	23	5	20	7	31	9	7	1
E	73	6	63	6	52	5	47	5	42	3
F	61	6	52	5	44	3	35	3	27	4
G	82	7	71	6	62	6	53	4	32	3

TABLE I.

The M.V.'s are also given to their nearest whole numbers. The first four subjects participated in the first part of the investigation. Thus they had only two exposure-series or fifty words each. The last three had four exposure-series or 100 exposure-words each. It will be noticed that the M.V.'s are not small enough to guarantee a good average in each case. Still, considering the complex nature of the operating factors in such experiments, they are not abnormally large. The M.V.'s in Washburn's experiment too were larger than 10% of the averages as calculated from the figures given in the published report of the experiment.

(2) The figures given below indicate the difference between the averages of the time required for the loss of meaning to set in the case of phonetically and non-phonetically spelled words.

Subject.	4-let.	5-let.	6-let.	8-let.
A	5	4	4	6
B	3	6	4	1
C	2	0	2	3
D	4	6	5	5
E	3	2	3	4
F	3	4	3	5
G	2	3	0	3

TABLE 2.

It will be seen that the difference in most cases forms, but a small percentage of the total average as given in Table 1. Thus, it might be said that the phonetically and the non-phonetically spelled words differ but little for the purposes of this experiment.

(3) The following table represents the variation in time due to the difference of space-interval. The time is given in seconds to the nearest whole numbers.

Space Int.	E	F	G
2 mm.	43		44
4 mm.	27	23	31

TABLE 3.

(4) Introspective data :—

(i) Certain visual effects appeared very frequently during the period of fixation. All the subjects have noticed these phenomena :—

(*a*) In the first place, the space between the letters either enlarges and the letters stand apart from one another; or it diminishes so that the letters approach one another and assume the shape of diphthong syllables. In both of these cases, the loss of meaning sets in.

(*b*) Secondly, the letters, especially *k*, *t*, *l*, etc., assume a slanting posture.

(*c*) Thirdly, the letters appear on different spatial planes.

(*d*) Fourthly, there is a peculiar spatial distortion of certain letters: *p*, *b*, *d*, etc., appear turned upside down. This has also been noticed by the subjects of Washburn's experiment.

(*e*) Lastly, there is a confusion of frequent occurrence between, *g* and *q*, *l*, and *i*, etc. This confusion is of a visual character and "seems to be the outcome of imperfect fusion between the image of the letters *q*, *i*, etc., and the percepts of *g*, *l*, etc."

(ii) Certain sensations of tension especially from the region of eye were often noticed.

(iii) Certain letters stand clearer to attention than other letters, e.g. *d*, *t*, and the first letter of the word. In those cases where these 'prominent' letters are in majority, the other letters are not noticed often, only that part of a word in which these prominent letters occur, is in the focus of attention; the other constituent letters are absolutely ignored.

(iv) The rôle of associational processes in connection with this phenomenon appears to be very important. Perception of any part of a word, even of single letters, appears to give rise to a number of associated images and ideas. It was almost impossible for the subjects to inhibit the associational processes. At the same time, associations in their turn, determine the groupings of letters into certain syllables. The loss of meaning was found to set in along with the associational processes. At times, the total word after losing the meaning for a time, regained it as the associations appeared.

(*v*) Sound images were not often noticed by the subjects. The conclusion reached by Washburn, thus, is not borne out by the records of this experiment. The differences between the averages of the phonetically and non-phonetically spelled letters as given in Table 2, also go to show that the auditory motor factor is not important

Conclusion.

(1) The visual effects very largely set in through the changes in accommodation, eye-movement, fatigue and adaptation. Thus, certain syllables are formed out of the letters of the word quite automatically.

(2) The space-interval between the letters enhances this effect.

(3) The larger the number of letters, the easier it is to form groups; and the larger the possible number of groups.

(4) Associations arise in connection with these visual groups.

(5) The loss of meaning is brought about through the shifting of attention from the visually presented word to the smaller artificial groups and to the associated images and ideas. The larger the number of the possible syllables, the greater is the chance of shifting of attention.

(6) The phenomenon under investigation is thus a product of two factors: (*i*) the visual changes and (*ii*) the simultaneously occurring associational processes. It is brought about through the shifting of attention from the total word to one of these.

ADDITIVE AND CONDENSATION PRODUCTS OF TRINITRO-*m*-CRESOL.

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AND

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Although our knowledge of the molecular compounds formed by picric acid and *s*-trinitrobenzene with other organic substances is fairly extensive, little work has so far been done on similar compounds which may be formed by the analogous trinitro-*m*-cresol. Noelting and Salis* prepared the additive compound of naphthalene with trinitro-*m*-cresol. The object of the present investigation is to make a systematic study of the additive compounds of trinitro-*m*-cresol with various classes of organic compounds.

The results of the investigation show that the trinitro-*m*-cresol like other poly-nitro aromatic compounds readily form coloured additive compounds with aromatic hydrocarbons, phenols, different classes of aliphatic and aromatic amines and their derivatives. These compounds have generally been prepared by mixing together the two constituents dissolved in suitable solvents. The additive products are generally very stable towards most solvents and some of them can be crystallised even from hot glacial acetic acid without any decomposition. These are all coloured and possess very beautiful crystalline structure. The results obtained may be summarised as follows:—

Hydrocarbons of the benzene series do not seem to yield any additive compounds. While naphthalene yields a yellow additive compound, acenaphthene, retene and fluorine yield deep yellow additive products. Diphenyl, dibenzyl and triphenyl-methane do not appear to combine with trinitro-*m*-cresol.

Phenols of the benzene series such as phenol, cresol, catechol, resorcinol and pyrogallol seem to form no additive compounds with trinitro-*m*-cresol. Only quinol seems to combine with trinitro-*m*-cresol as shown by the intense red colouration of the solution when both are mixed in hot alcohol, but on cooling only trinitro-*m*-cresol separates out perhaps owing to decomposition. The phenols of the naphthalene series as *α*-and *β* naphthols yield orange-coloured additive products. The colour of the product with the latter is deeper than that of the product with the former.

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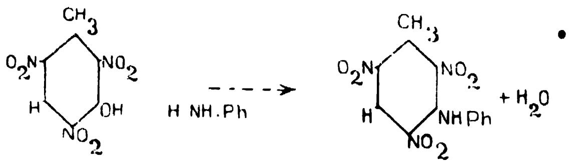
* *Ber.*, *15*, 1862 (1882).

Aliphatic amines and ammonium bases as allylamine, ethylene and trimethylene diamines and neurine hydrochloride yield deep yellow additive products with trinitro-*m*-cresol in molecular proportions of (1 : 1) excepting allylamine which forms an additive product with 1 mol. of the base combined to 3 mols. of trinitro-*m*-cresol. The cyclic base piperazine yields a deep yellow additive compound with trinitro-*m*-cresol in molecular proportions of one of the former to two of the latter.

It is very interesting to observe that the aliphatic amines hexyl and heptyl amines and aromatic amines as aniline, the three toluidines and nitrotoluidine ($CH_3 : NO_2 : NH_4 = 1 : 2 : 4$) yield light to deep yellow coloured compounds which are not mere additive products. This is shown by the fact that on analysis they give abnormal values, which can only be explained by assuming a sort of condensation to take place between the phenolic (OH) on the one hand and the hydrogen of the NH_2 group on the other, a molecule of water being removed, substituted trinitro-*m*-toluidines being formed thereby.



Such a reaction might be possible as trinitro-*m*-toluidine itself is prepared by the action of alcoholic ammonia on trinitro-*m*-cresol ethyl ether* and on 3-chloro† and 3-bromo‡-2,4,6-trinitro-*m*-toluene a molecule of alcohol being liberated in the first case and a molecule of halogen acid being liberated in the second and third.

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* Noelting and Salis, *Ber.*, *15*, 1864 (1882).

† F. Reverdine and A. Deletra, *Ber.*, *37*, 2094 (1904)

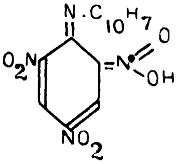
‡ *American Chemical Journal*, *12*, 4.

F. Reverdine and A. Deletra* prepared the compounds of the above type by boiling an alcoholic solution of trinitro chloro toluene with each of the bases aniline, *p*-toluidine, *p*-amidophenol and *p*-phenylenediamine, a molecule of hydrochloric acid being removed in each case. But while the condensation products prepared in this way with aniline and *p*-toluidine melt respectively at 150° and 127°, the corresponding compounds from trinitro-*m*-cresol with the same bases decompose between 169-171° and 175-179° respectively, showing that they are different from one another.

Again J. Sudborough and N. Picton† prepared an analogous series of condensation products in a similar way by boiling an alcoholic solution of picryl chloride with an excess of a base, a molecule of hydrochloric acid being removed in each case. The authors have found that this type of compounds form (1) a mono-potassium salt with methyl alcoholic KOH, (2) yield no acetyl derivative with acetic anhydride or acetyl chloride, (3) are very stable towards dilute or strong mineral acids, (4) form additive compounds with aryl amines and (5) some of them exist in two isomeric forms. But the analogous condensation products obtained from trinitro-*m*-cresol and the bases have none of these properties and seem to be decomposed when treated with the first three reagents showing that they belong to a different type.

If a conclusion might be ventured it cannot be denied that trinitro-*m*-cresol behaves differently from trinitrochloro-toluene in this respect and the difference should be one of a structural nature. This difference in structure can be explained in the following way:—

Sudborough‡ holds that the red variety of the condensation products of picryl chloride with aromatic bases such as the naphthylamines possess an ortho or para quinonoid structure and represents them as



Applying similar conception to Reverdin's§ compounds we can represent them in the following manner:—

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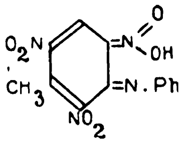
* Loc. cit.

† J. Sudborough and N. Picton, *J. Chem. Soc.*, 89, 583 (1906).

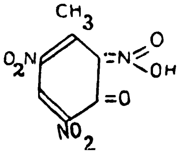
‡ Ibid.

§ Ibid.

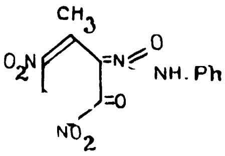


As these compounds differ from similar compounds obtained from trinitro-*m*-cresol the formation of the latter series of compounds may be supposed to take place differently.

According to Armstrong's views trinitro-*m*-cresol should have an *o*-quinonoid structure and in this form it reacts with the amine giving



rise to condensation products of the type



with the elimination of a molecule of water.

Although such a view of the constitution of the end products in the case of trinitro-*m*-cresol well explains their instability and the beautiful colour which most of them possess yet the constitution cannot be held as established and much work has to be done to clear this point.

In spite of their anomalous behaviour if these compounds be real condensation products they are no doubt remarkable as they show conclusively that additive compounds indicate the affinity which exists between the substances. In some cases the compounds remain side by side, in others chemical reaction actually takes place. However it is hoped that further investigation on the subject will throw much light on the exact nature of additive compounds in general.

A comparison of the other properties of these compounds show that a single methyl substituent in the arylamine molecule increases the stability and deepens the colour of the combined product especially when the substitution takes place in the para position. For example in the toluidines the colour deepens from the ortho through meta to the para compound. Benzylamine also forms a deep yellow additive compound.

Single negative substituents such as Cl, Br, I and NO_2 in the arylamine molecule in the *p*-position to NH_2 group do not prevent the formation of additive compounds. It is interesting that *p*-chloro-, *p*-bromo- and *p*-iodo-anilines form additive products with trinitro-*m*-cresol in the proportion of one molecule of the base to two molecules of the latter. While *m*- and *p*-nitroanilines form additive compounds with trinitro-*m* cresol, the ortho compound does not appear to yield any additive product in accordance with its behaviour with picric acid. The same effect of the NO_2 group in the *o*-position to NH_2 group is seen in 1, 3, 4-nitrotoluidine ($CH_3 : NO_2 : NH_2 = 1 : 3 : 4$). These cases are perhaps due to steric hindrance and further supports indirectly the view that the phenolic (OH) of trinitro-*m*-cresol has an attraction for (NH_2) group at least in the additive compounds of the amines.

The introduction of another NH_2 group into the arylamine molecule as in *m*-phenylene diamine deepens the colour of the additive product. The amines of the naphthalene series as naphthylamine form more deeply coloured and more stable additive product than the amines of the benzene series. Camphyl and menthylamines form deep yellow coloured additive products in the proportion of one molecule of the base combined with two molecules of trinitro-*m*-cresol.

The formation of these characteristic derivatives with trinitro-*m*-cresol can be used for the detection of small quantities of various amines and it is highly probable that the compounds will prove of use in the purification of a number of different amines, as most of them crystallise remarkably well, are readily obtained in a state of purity and can in many cases be decomposed by mineral acids.

EXPERIMENTAL.

PREPARATION OF 2, 4, 6-TRINITRO-*m*-CRESOL.

2, 4, 6-trinitro-*m*-cresol has been obtained in good yield by adopting the principle of Noelting's method* with some modifications.

A weighed quantity of *m*-cresol was taken in a round-bottomed flask and 3 times its weight of conc. H_2SO_4 was added to it. It was heated for 10 to 15 minutes on the water-bath and allowed to stand for 3 to 4 days in a warm place. Before nitration it is tested whether any cresol separates out on dilution with water. When the sulphonation is complete the cresol sulphonic acid is diluted by adding twice its volume of cold water. The nitration is commenced by adding small quantities of dil. HNO_3 and the mixture is heated on the water-bath. When the first violent reaction is over, more conc. HNO_3 is added till the nitration is complete. Then it is poured into a basin and evaporated on the water-bath to about half its volume. After some time practically all the trinitro-*m*-cresol separates out in flakes on the surface. These are secured by suction. The filtrate is further evaporated when a further crop of crystals is obtained. These are drained at the pump.

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* Noelting and Salis, *Ber.*, *14*, 987 (1881).

The crude trinitro-*m*-cresol thus obtained is then boiled with a small quantity of water on the water-bath in order to free it from a little oxalic acid and picric acid. It is then freed from adhering liquid by suction and recrystallised from hot water. It is obtained in fine pale yellow needles melting at 105°.

ADDITIVE COMPOUNDS.

ACENAPHTHENE-TRINITRO-*m*-CRESOLATE.

Saturated hot benzene solutions of equimolecular quantities of acenaphthene and trinitro-*m*-cresol were mixed together when the colour of the mixed solution became deep red. On cooling a yellow product was obtained. This on recrystallisation from absolute alcohol melts sharply at 120 . It has been found to be acenaphthene-trinitro-*m*-cresolate.

0·0363 gave 4·3 c.c. N_2 at 30 and 758 mm., N = 13·04

Calc. for $C_{12}H_{10},2C_6H.(CH_3)(OH)(NO_2)_3$; N = 13·13

The substance is easily soluble in alcohol and benzene. Water seems to decompose the substance into its components even when the former is present in small quantities in alcohol. The cresolate is obtained in yellow feathery needles from alcohol.

FLUORENE-TRINITRO-*m*-CRESOLATE.

Hot saturated benzene solutions of fluorene and trinitro-*m*-cresol were mixed together in molecular proportions and the deep red coloured solution thus obtained was allowed to cool slowly. A yellow crystalline product was thus obtained which on crystallisation twice from hot benzene melts sharply at 109°. This has been found to be fluorene-trinitro-*m*-cresolate.

0·1049 gave 10·2 c.c. N_2 at 30° and 753 mm.; N = 10·64

Calc. for $C_{13}H_{10}, C_6H.(CH_3)(OH)(NO_2)_3$; N = 10·27

It is easily soluble in ether, alcohol and benzene, but insoluble in water and is decomposed by warm aqueous alcohol. It is best recrystallised from benzene from which long deep yellow needles are obtained.

ALLYLAMINE-TRINITRO-*m*-CRESOLATE.

Hot saturated solutions of allylamine and trinitro-*m*-cresol in benzene, the former being taken in excess, were mixed together when a yellow product was at once formed which is soluble in hot benzene mixed with a little allylamine. By spontaneous evaporation of the benzene solution yellow cubical plates were obtained from which traces of allylamine were removed by drying on a porous plate and subsequently by washing the crystals twice by means of hot benzene. It was finally recrystallised from a mixture of benzene and alcohol when a product melting sharply at 165° was obtained. This has been found to be allylamine-trinitro-*m*-cresolate.

0·0556 gave 9·9 c.c. N_2 at 23·5° and 766 mm.; N = 20·32

0·0567 gave 10·1 c.c. N_2 at 24° and 765 mm.; N = 20·27

Calc. for $3C_3H_7N, C_6H.(CH_3)(OH)(NO_2)_3$; N = 20·27

The cresolate is insoluble in ether and benzene. It is soluble moderately in water and highly in alcohol and in glacial acetic acid. The most suitable solvent for it is a mixture of alcohol and benzene from which the crystals of the cresolate are obtained in shining yellow cubical plates.

BENZYLAMINE-TRINITRO-*m*-CRESOLATE.

To a hot saturated aqueous solution of benzylamine hydrochloride a saturated aqueous solution of trinitro-*m*-cresol was added and the mixture was allowed to cool slowly. After some time small golden yellow needles separated. It was recrystallised twice from hot water when it melts sharply at 184–185°. This has been found to be benzylamine-trinitro-*m*-cresolate.

0·1206 gave 17·4 c.c. N_2 at 30° and 760 mm.; N = 15·93

Calc. for $C_7H_9N. C_6H.(CH_3).(OH)(NO_2)_3$; N = 16·00

It is easily soluble in water and in alcohol. On recrystallisation from hot water small golden yellow needles were obtained which melted sharply at 184–185°.

ETHYLENEDIAMINE-TRINITRO-*m*-CRESOLATE.

A hot saturated aqueous solution of the hydrochloride of the base and an aqueous alcoholic solution of trinitro-*m*-cresol were mixed together in equimolecular proportions. On cooling small yellow needles were obtained which on recrystallisation twice from hot water decomposes at 200–204°. The product has been found to be ethylenediamine-trinitro-*m*-cresolate.

0·0524 gave 10·6 c.c N_2 at 26° and 763 mm.: N = 22·78

Calc. for $C_2H_8N_2 . C_6H(.CH_3)(OH)(NO_2)_3$; N = 23·10

The cresolate is soluble in hot water and in alcohol. It consists of small shining lemon yellow needles which decompose slowly above 200°, the yellow substance turning brown. The product decomposes completely at 204° with frothing and charring.

TRIMETHYLENEDIAMINE-TRINITRO-*m*-CRESOLATE.

Trimethylenediamine and trinitro-*m*-cresol were weighed out in equimolecular proportions, the former being taken in a little excess. Saturated solutions of both in ether were mixed together. A yellow product was formed at once and this on recrystallisation from hot water was obtained as a pure substance melting with decomposition. This has been found to be trimethylenediamine-trinitro-*m*-cresolate.

0·0545 gave 10·6 c.c. N_2 at 26° and 763 mm.; N=21·91

Calc. for $C_3H_{10}N_2, C_6H.(CH_3)(OH)(NO_2)_3$; N=22·08

The cresolate is soluble slightly in ether, moderately in water and easily in alcohol. It consists of yellow granular crystals and decomposes slightly above 200°, the decomposition increasing with temperature and being completed between 205–210°. It melts with charring and frothing at 213°.

CAMPHYLAMINE-TRINITRO-*m*-CRESOLATE.

A hot saturated solution of trinitro-*m*-cresol in water was mixed with a little excess of camphylamine dissolved in dilute HCl. The mixture on cooling yielded a product which on recrystallisation several times from hot water melts sharply at 183°. This has been found to be camphylamine-trinitro-*m*-cresolate.

0·0648 gave 8·7 c.c. N_2 at 23° and 766 mm.; N=15·37

0·0836 gave 11·1 c.c. N_2 at 26° and 763 mm.; N=14·95

Calc. for $C_{10}H_{19}N \,.\, 2C_6H \,.\, (CH_3)(OH)(NO_2)_3$; N=15·33

The cresolate is moderately soluble in hot water and in alcohol, but the substance seems to decompose slightly in the latter solvent. The product is obtained as fine lemon yellow needles from hot water.

MENTHYLAMINE-TRINITRO-*m*-CRESOLATE.

Saturated hot aqueous solutions of the hydrochloride of menthylamine and trinitro-*m*-cresol, the former being taken in little excess were mixed together and was allowed to cool slowly when long yellow needles were deposited. This on recrystallisation from hot water melts at 204° with decomposition. This has been found to be menthylamine-trinitro-*m*-cresolate.

0·0710 gave 9·5 c.c. N_2 at 22·5° and 766 mm.; N=15·34

Calc. for $C_{10}H_{21}N,\ 2C_6H \,.\, (CH_3)(OH)(NO_2)_3$; N=15·29

The cresolate is highly soluble in alcohol, but very slightly in cold water. The product is obtained as long lemon yellow shining needles from hot water. It decomposes slightly above 200°C. and melts with charring and frothing at 204°C.

NEURINE HYDROCHLORIDE TRINITRO-*m*-CRESOLATE.

A saturated solution of one gram of trinitro-*m*-cresol in aqueous alcohol was mixed with a saturated solution of one gram of neurine hydrochloride in water, both in the hot state. On cooling, long lemon yellow needles separated, which on recrystallisation twice from hot water melts at 162°. This has been found to be neurine hydrochloride trinitro-*m*-cresolate.

0·0646 gave 9 c.c. N_2 at 32° and 756 mm.; N=15·17

Calc. for $C_5H_{12}NCl \,.\, C_6H \,.\, (CH_3)(OH)(NO_2)_3$; N=15·36

This is soluble in water and alcohol.

PIPERAZINE-TRINITRO-*m*-CRESOLATE.

A saturated solution of more than the calculated amount of the hydrochloride of the base piperazine in hot water was mixed with a hot saturated aqueous solution of trinitro-*m*-cresol. On cooling deep yellow needles separated from the mixed solution, which on recrystallisation from a hot mixture of alcohol and water (1 : 3) gave a pure product decomposing at 220–225°. This has been found to be piperazine trinitro-*m*-cresolate.

0·0366 gave 7·8 c.c. N_2 at 33°C. and 759·5 mm.; N=23·28
0·0691 gave 15 c.c. N_2 at 31°C. and 757·5 mm.; N=23·79
Calc. for $2C_4H_{10}N_2$, $C_6H.(CH_3)(OH)(NO_2)_3$; N=23·61

The cresolate is moderately soluble in water and readily in alcohol. It consists of fine lemon yellow shining needles. The product decomposes between 220–225°. The decomposition however begins slowly above 200°, the colour of the substance changing from deep yellow to brown.

m-PHENYLENEDIAMINE-TRINITRO-*m*-CRESOLATE.

m-Phenylenediamine hydrochloride and trinitro-*m*-cresol were weighed out in equimolecular proportions and saturated aqueous solutions of both were warmed in a round-bottomed flask. The mixture was then allowed to cool very slowly from which a dirty brown product separated on cooling It was recrystallised from hot water until a pure product was obtained. This has been proved to be *m*-phenylenediamine trinitro-*m*-cresolate

0·0846 gave 15·3 c.c. N_2 at 31° and 754 mm., N=19·73
Calc. for $C_6H_4(NH_2)_2$, $C_6H.(CH_3)(OH)(NO_2)_3$; N=19·94

The cresolate is soluble in water and alcohol. Dirty brown shining plates are obtained from hot water, which melt at 166° with sudden frothing and charring.

a-NAPTHYLAMINE-TRINITRO-*m*-CRESOLATE.

The hydrochloride of the base was first prepared by dissolving the base in just sufficient amount of dilute HCl and warming. Then a hot saturated solution of trinitro-*m*-cresol was added to it. Dirty yellow needles separate on cooling, which on recrystallisation thrice from hot water melt with decomposition. This has been found to be -naphthylamine-trinitro-*m*-cresolate.

0·0536 gave 7·3 c.c. N_2 at 32° and 755 mm.; N=14·81
0·0771 gave 10 c.c. N_2 at 30° and 754 mm.; N=14·20
Calc. for $C_{10}H_9N.C_6H.(CH_3)(OH)(NO_2)_3$; N=14·51

The cresolate is soluble in water and alcohol. The pure product melts with decomposition between 165–170°, the volume suddenly expanding at 170°.

p-CHLORANILINE-TRINITRO-*m*-CRESOLATE.

Equimolecular quantities of *p*-chloraniline and trinitro-*m*-cresol dissolved in benzene were mixed together, when combination took place vigorously with evolution of heat and a yellow product separated out. This on recrystallisation from hot water melts at 170–172°. It has been found to be *p*-chloraniline-trinitro-*m*-cresolate.

0·0696 gave 10 c.c. N_2 at 26° and 760 mm.; N = 16·12
Calc. for $C_6H_6NCl \,.\, 2C_6H \,.\, (CH_3)(OH)(NO_2)$; N = 15·98

The product is very slightly soluble in benzene, insoluble in ether, fairly soluble in water and readily in alcohol. The cresolate is obtained in beautiful fine lemon yellow sponzy needles from water.

m-NITRANILINE-TRINITRO-*m*-CRESOLATE.

Saturated hot aqueous solutions of *m*-nitraniline and trinitro-*m*-cresol were mixed together in molecular proportions by weight and the mixture was then allowed to cool slowly when a yellow product appeared on cooling. This was recrystallised thrice from hot water. The pure product has been found to be *m*-nitraniline-trinitro-*m*-cresolate.

0·1188 gave 20·6 c.c. N_2 at 32° and 751 mm.; N = 18·76
Calc. for $C_6H_6N_2O_2,\ C_6H \,.\, (CH_3)(OH)(NO_2)_3$; N = 18·37

The cresolate is obtained as long yellow needles melting sharply at 114°. It is moderately soluble in water and easily in alcohol.

p-NITRANILINE-TRINITRO-*m*-CRESOLATE.

Hot saturated aqueous solutions of equimolecular quantities of *p*-nitraniline and trinitro-*m*-cresol were mixed together. The mixture was allowed to cool slowly. The product thus obtained on recrystallisation twice from hot water and finally from a mixture of water and alcohol (3 : 1) melts sharply at 103°. This has been found to be *p*-nitraniline-trinitro-*m*-cresolate.

0·0638 gave 10·9 c.c. N_2 at 32° and 755 mm.; N = 18·58
Calc. for $C_6H_6N_2O_2,\ C_6H \,.\, (CH_3)(OH)(NO_2)_3$; N = 18·37

It consists of long deep yellow needles soluble both in water and alcohol, but it seems to decompose slightly when it is boiled with water for a long time.

α-NAPHTHOL-TRINITRO-*m*-CRESOLATE.

Saturated solutions of α-naphthol and trinitro-*m*-cresol in hot aqueous alcohol were mixed together in molecular proportions. The colour of the solution changes to deep brown and on cooling orange coloured needles crystallised out. The product was recrystallised from aqueous alcohol when it melts at 159°. This has been found to be α-naphthol-trinitro-*m*-cresolate.

0·1288 gave 13·4 c.c. N_2 at 30° and 754 mm.; N = 11·24

Calc. for $C_{10}H_8O$, $C_6H.(CH_3)(OH)(NO_2)_3$; N = 10·85

These orange yellow coloured silky needles are slightly soluble in water and easily in alcohol. The pure product melts sharply at 159°.

β-NAPHTHOL TRINITRO-*m*-CRESOLATE.

Molecular proportions of β-naphthol and trinitro-*m*-cresol in hot saturated aqueous alcoholic solutions were mixed together and was then allowed to cool slowly. After some time orange coloured needles were deposited which on recrystallisation twice from aqueous alcohol melt sharply at 124°. This has been found to be β-naphthol-trinitro-*m*-cresolate.

0·0685 gave 7·2 c.c. N_2 at 30° and 753 mm.; N = 11·42

0·0663 gave 7·1 c.c. N_2 at 34·5° and 753 5 mm.; N=11·48

Calc. for $C_{10}H_8O$, $C_6H.(CH_3)(OH)(NO_2)_3$; N=10·85

It is moderately soluble in water and readily in alcohol. It is also soluble in benzene and ether. It consists of orange coloured silky needles, melting to a red liquid.

p-IODOANILINE TRINITRO-*m*-CRESOLATE.

Hot aqueous alcoholic solutions of *p*-iodoaniline and trinitro-*m*-cresol were mixed together the former being taken in excess. On cooling a yellow precipitate was thrown out, on recrystallising which twice from hot aqueous alcohol pure yellow needles were obtained melting sharply at 152°. It has been found to be *p*-iodoaniline-trinitro-*m*-cresolate.

0·0913 gave 11·4 c.c. N_2 at 34° and 761 mm.; N=13·59

Clac. for C_6H_6NI, $C_6H(CH_3)(OH)(NO_2)_3$; N = 13·90

It consists of fine bright yellow needles melting sharply at 152°, to a black liquid. It is slightly soluble in water, but easily in alcohol and benzene.

p-BROMANILINE TRINITRO *m*-CRESOLATE.

To a hot saturated solution of trinitro-*m*-cresol in dilute alcohol, a hot saturated solution of *p*-bromaniline in water was added, the latter being taken in a little excess. On cooling slowly a yellow product separated which on being recrystallised from hot aqueous alcohol gave a pure product melting sharply at 172·5°. This has been found to be *p*-bromaniline trinitro-*m*-cresolate.

0·0593 gave 8 c.c. N_2 at 30° and 763 mm.; N=14·95
Calc. for $C_6H_6NBr, 2C_6H(CH_3)(OH)(NO_2)$; N=14·89

It consists of fine lemon yellow granular crystals. It is soluble slightly in water and benzene but readily in alcohol, acetone and ether. The pure product melts to a dark-brown liquid sharply at 172·5°.

CONDENSATION PRODUCTS OF TRINITRO-*m*-CRESOL WITH ANILINE.

A saturated hot aqueous solution of aniline hydrochloride is mixed with a hot aqueous solution of trinitro-*m*-cresol both being taken in equimolecular quantities. The mixed solution is heated in a round-bottomed flask. On cooling, yellow crystals were obtained. The product was recrystallised twice from hot water till a pure product with constant decomposition temperature was obtained. Its analytical results can be explained by assuming it to belong to the type of substituted trinitro-*m*-toluidines.

0·0807 gave 13·0 c.c. N_2 at 30° and 761 mm.; N=17·80
0·1099 gave 0·1985 of CO_2 and ·0551 H_2O; C=49·24; H=5·57
Calc. for $C_6H . CH_3(NO_2)_3 . NH.C_6H_5$; C=49·06; H=3·14;
N=17·61.

It is soluble in water and alcohol. It consists of long shining yellow needles. The pure product decomposes slightly above 166° and completely between 169–171° with frothing. It is decomposed by acetic anhydride, strong and dilute mineral acids and methyl alcoholic potash. It is different from phenyl-trinitro-*m*-toluidine prepared by Reverdine.*

HEPTYLAMINE.

A hot saturated aqueous solution of the hydrochloride of the base was added to a saturated aqueous alcoholic solution of trinitro-*m*-cresol both being taken in equimolecular quantities. The mixture was heated in a round-bottomed flask and the clear solution on cooling slowly deposited long yellow crystals. It was recrystallised from aqueous alcohol till a pure product melting at 159° was obtained. Its analysis suggests it to be heptyl-trinitro-*m*-toluidine.

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* F. Reverdine and A. Deletra, *Ber.*, *37*, 2094.

0·0774 gave 11·4 c.c. N_2 at 25° and 763 mm. ; N = 16·66
0·0909 gave 0·1692 of CO_2 and 0·0583 H_2O ; C=50·76 ; H = 7·12
Calc. for $C_7H_4(NO_2)_3 . NH . C_7H_{15}$; N = 16·47, C = 49·41, H=6·06.

It is soluble slightly in water and easily in alcohol and benzene. It consists of long shining lemon yellow crystals melting sharply at 159° to a red liquid. It is decomposed by acetic anhydride, strong and dilute mineral acids.

o-TOLUIDINE.

Hot saturated aqueous solutions of equimolecular quantities of *o*-toluidine and trinitro-*m*-cresol were mixed together, the base being converted into its hydrochloride. The solution was heated to boiling. The clear solution on cooling deposited long needles. It was recrystallised twice from hot water till a pure product was obtained. Its analysis can be explained by assuming the formation *o*-tolyl-trinitro-*m*-toluidine.

0·1027 gave 15·3 c.c. N_2 at 28° and 764·5 mm. ; N = 16·68
0·0848 gave 12·8 c.c. N_2 at 27° and 761 mm. ; N = 16·88
0·1228 gave 0·2141 of CO_2 and 0·0559 H_2O ; C=47·54, H = 5 058
0·1332 gave 0·2336 of CO_2 and 0·0534 H_2O ; C=47·83 ; H = 4·45
Calc. for $C_7H_4(NO_2)_3 . NH . C_7H_7$; C=50·60 ; H=3·61 ; N=61·86

It is soluble in water and alcohol. It consists of long light yellow needles. The pure product decomposes slightly above 165° and melts with charring and frothing at 171°C. It is unstable towards acetic anhydride and dilute and strong mineral acids.

m-TOLUIDINE

When a hot aqueous solution of the hydrochloride of the base was added to an aqueous solution of an equimolecular quantity of trinitro-*m*-cresol, a yellow product was obtained on cooling, which was purified by recrystallisation from hot water. The analysis of the pure product is explained by assuming the formation of the condensation product *m*-tolyltrinitro-*m*-toluidine.

0·0691 gave 10·6 c.c. N_2 at 29° and 762 mm.; N = 17·05

0·0808 gave 0·1481 of CO_2 and 0·0379 H_2O; C=50·00, H = 5·21

Calc. for $C_7H_4(NO_2)_3$. NH. C_7H_7; C=50·60; H=3·61; N=16.86

It consists of deep yellow shining needles soluble in water and alcohol. The pure product melts with charring and frothing between 170–173°C. It is decomposed by dilute and strong mineral acids and acetic anhydride.

p-TOLUIDINE.

Saturated aqueous alcoholic solutions of the free base and trinitro-*m*-cresol in equimolecular quantities were mixed together in the hot. The solution was heated to boiling. On cooling a yellow product was obtained. On crystallising it from aqueous alcohol long deep yellow needles were obtained which completely decompose at 179°. Its analytical results can be explained by assuming it to belong to the type of substituted trinitro-*m*-toluidines.

0·0957 gave 14·9 c.c. N_2 at 32° and 761 mm.; N = 17·08

0·1006 gave 0·1748 of CO_2 and 0·0448 H_2O; C=47·40; H = 4·95

0·1142 gave 0·1988 CO_2 and 0·0480 H_2O; C=47·50; H=4·67

Calc. for $C_6H(CH_3)(NO_2)_3$. NH. C_6H_4. CH_3; N=16·86, C=50·60; H = 3·61

It consists of long deep yellow needles. It is soluble in water, alcohol and benzene. The pure product decomposes slightly above 171° and completely between 175–179° with charring and frothing at 179°. It is different from *p*-tolyltrinitro-*m*-toluidine.* It is unstable towards mineral acids, acetic anhydride and methyl alcoholic potash.

NITROTOLUIDINE. ($CH_3 : NO_2 : NH_2 = 1 : 2 : 4$)

A saturated hot aqueous solution of 1, 2, 4-nitro-toluidine was added to a hot saturated aqueous solution trinitro-*m*-cresol both substances being taken in equal proportions by weight. On cooling a yellow granular precipitate was thrown down which was obtained pure by recrystallising it twice from hot water. The pure product melts sharply at 146·5°. Its analytical results agree with the formula of the condensation product 1:2: 4-nitrotolyl-trinitro-*m*-toluidine.

0·0621 gave 10·4 c.c N_2 at 26° and 761 mm.; N=18·80

0·1112 gave 18·0 c.c. N_2 at 26° and 761 mm.; N=18·20

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* F. Reverdine and A. Deletra, *Ber.*, *37*, 2094.

0·1196 gave 0·1968 CO_2 and 0·0471 H_2O ; C=44·87; H=4·37
Calc. for $C_6H(CH_3)(NO_2)_3 . NH . C_6H_3(CH_3)(NO_2)$; N=18·57,
C=44·56, H=2·92

It is soluble moderately in water, highly in alcohol, benzene and glacial acetic acid. From the last mentioned solvent the crystals are obtained by the spontaneous evaporation of the solvent on a porous plate. It consists of granular crystals with a yellowish tinge melting sharply at 146·5°. It is not stable towards mineral acids and acetic anhydride.

HEXYLAMINE.

Equimolecular quantities of hexylamine hydrochloride and trinitro-*m*-cresol are weighed out and the hot saturated aqueous solution of the former was added to the hot saturated solution of the latter in aqueous alcohol. The solution was heated to boiling. The clear solution on cooling deposited long yellow needles. It was recrystallised twice from hot aqueous alcohol till a pure product melting sharply at 159° was obtained. Its analytical results agree with the formula hexyltrinitro-*m*-toluidine.

0·0995 gave 14·95 c.c. N_2 at 25° and 765 mm.; N=17·04
0·1276 gave 0·2222 CO_2 and 0·0784 H_2O ; C=47·50, H=6·83
Calc. for $C_6H(CH_3)(NO_2)_3 . NH. C_6H_{13}$; N=17·17, C=47·85,
H=5·52

It is soluble in water and alcohol. It consists of long lemon-yellow shining needles, melting sharply at 159° to a red liquid. It is decomposed by mineral acids, dilute or strong, and by acetic anhydride.

THE ACOUSTICAL KNOWLEDGE OF THE ANCIENT HINDUS.

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1. Introduction.

Music, both vocal and instrumental, undoubtedly played an important part in the cultural life of ancient India. Sanskrit literature, both secular and religious, makes numerous references to instruments of various kinds, and it is, I believe, generally held by archaeologists that some of the earliest mentions of such instruments to be found anywhere are those contained in the ancient Sanskrit works. Certain it is that at a very early period in the history of the country, the Hindus were acquainted with the use of stringed instruments excited by plucking or bowing, with the transverse form of flute, with wind and reed instruments of different types and with percussion instruments. It is by no means improbable that India played an important part in the progressive evolution and improvement of these instruments and might have served as a source from which their knowledge spread both eastwards and westwards. It would form a fascinating chapter of history to try and trace the gradual development of musical instruments and musical knowledge, from the rhythmic chanting of the Ṛgveda in the ancient home of the Aryan race to the Indian music of the present day. But the materials available for the writing of this history seem to be all too meagre. Much of the long period over which the gradual evolution must have spread lies in the dim and remote past of which but the vaguest glimpses can be obtained from such records as exist. Something more definite regarding the acoustical developments in Ancient India might perhaps be gleaned from a study of the musical instruments, the models of which have been handed down as heirlooms for untold generations. Several of the Indian stringed instruments, for example, disclose in their design, even on a superficial examination, a quite remarkable appreciation of the principles of sound-production and of resonance. A fuller study seemed likely to lead to results of considerable interest. It was this hope that induced me some two years ago to commence a systematic examination by modern scientific methods of the ancient Indian musical instruments. The objects I set before myself were to investigate the traditional designs according to which these Indian instruments are constructed and the variations of these designs that exist in the different parts of the country, to discover the *raison d'être* of the methods of construction employed and to find the special tone-characters which were held in esteem by the designers. It seemed that such an examination might also prove useful from the practical stand-point by disclosing the best designs and indicating the directions in which any improvements might be possible. Various circumstances have delayed the complete carrying out of the projected work, and it is probable that little progress might have been made with it up to date, but for the fact that my attention was recently drawn somewhat forcibly to the musical qualities of the ancient Indian instruments of percussion. Through the kindness of an enthusiastic fellow-worker, Mr. Sivakali Kumar, some good specimens of the Indian percussion instruments were put at my disposal and I have been enabled to carry out a scientific examination of their acoustical properties. The results obtained are very remarkable and significant and are being described in detail in a monograph "On Musical Drums" which will be published by the Indian Association for the Cultivation of Science. I propose in this short essay to indicate the main results of this investigation and to show how far they throw light on the state of acoustical knowledge in ancient and mediaeval India.

2. *Acoustics of Percussion Instruments.*

By way of preface, I shall first refer to a few facts regarding the vibrations of stretched membranes which are familiar to students of physics and which it is useful here to recall. As is well known, the vibrations of a circular stretched membrane or drum-head excited by impact are generally of an extremely complex character. Besides the gravest or fundamental tone of the membrane, we have a large retinue of overtones which stand to each other in no sort of musical relation. These overtones are always excited in greater or less degree and produce a discordant effect. All the instruments of percussion known to European physicists in which a circular drum-head is employed have therefore to be regarded more as noise producers introduced for marking the rythm than as musical instruments. This is true even of the kettle-drum which is tuned to a definite pitch and occasionally used in European orchestral music. As has been shown by the late Lord Rayleigh in a paper published some time ago, the air enclosed in the shell of the kettle-drum does not produce any advantageous alterations of the pitch relations of the overtones. All the instruments of percussion known to European science are thus essentially non-musical and can only be tolerated in open air music or in large orchestras where a little noise more or less makes no difference. Indian musical instruments of percussion however stand in an entirely different category. Times without number we have heard the best singers or performers on the flute or violin accompanied by the well-known indigenous musical drums, and the effect with a good instrument is always excellent. It was this, in fact, that conveyed to me the hint that the Indian instruments of percussion possess interesting acoustic properties, and stimulated the research.

3. *The Indian Musical Drums.*

The number of different types of percussion instruments known and used in India is almost legion. They represent a very wide variety of stages of development and variations of form to suit different purposes. It does not fall within the scope of this short essay even to attempt a discussion of the different forms. Those who are curious to see these types of drums can no doubt find specimens in the various provincial museums of India. A specially good collection is to be found in the anthropological section of the Indian Museum at Calcutta, and some of them are described with illustrations in the catalogue of the exhibits available in the Museum. The instrument to the remarkable acoustical properties of which I wish especially to direct attention is the musical or concert drum, which is most highly esteemed by Indians and which figures largely in the Sanskrit literature, namely the *Mṛdaṅga*. The essential feature of this instrument at the present day is, first, a massive hollow wooden body in the form of two truncated cones put end to end, one of which is longer than the other. Over the two ends of this body are stretched the two drumskins, which are each provided with a tightening ring of leather and are kept in a state of tension by a leather rope which passes through apertures in the rings at 16 equidistant points around the circumference. Eight cylindrical tuning blocks of wood inserted at regular intervals under the tension-rope provide the means for a rough adjustment of tension. The fine adjustment of tension of the smaller drumhead to equality in the 8 octants of the circumference is carried out by pulling up or pushing down the tightening ring by stroking it with a small mallet. The large drumhead gives the base note, and its pitch and tone-quality are adjusted by spreading a temporary load of wetted *āṭā* or wheaten flour over it. The most remarkable feature of the drum is the manner in which the second or smaller drumhead is constructed. This membrane as first put on in the construction of the drumhead is double, the layers being of specially chosen leather of uniform thickness and connected to the tightening ring so as to be in a state of tension. The upper layer is then cut away in the middle exposing a circular area of the lower membrane, and leaving an annular ring of the outer membrane round the margin, of which the width is regulated according to the requirements of the tone-quality. The centre of the exposed circle of the inner membrane thus formed is loaded concentrically in several successive layers of gradually decreasing radii and of graduated thickness with a dark coloured composition which is put on at first in the form of a paste and is then rubbed in till it becomes dry and permanently adherent to the membrane. The composition of this material is finely powdered iron filings, charcoal and starch, and when put on the membrane it is flexible in a noteworthy degree. The putting on of the load is carried out in stages, the sound of the drumhead being continuously tested during the progress of construction. Its final adjustment and regulation of thickness is an art which is handed down from generation to generation as traditional knowledge, and acquired by long training and experience.

4. *The Acoustic Characters of the Mṛdaṅga.*

A physicist trained in acoustical research noticing the drumhead of the *Mṛdaṅga* naturally wishes to know exactly what acoustical purpose is intended to be served by the peculiar method of construction described above This is a question which can only be answered by a physical examination of the vibrations of the drum-head and of the tones to which it gives rise. Such an examination has been carried out by me and has led to extremely remarkable results. It was stated above that a percussion instrument generally gives rise to inharmonic overtones. The examination of the *Mṛdaṅga* shows that it forms an exception to this rule, and gives rise to harmonic or musical overtones in the same manner as a stringed instrument. I find in fact that the physical behaviour of the drumhead is in many respects unlike that of an ordinarily circular stretched membrane, and approaches that of a stretched string. In the same manner as a stretched string, the loaded membrane of the *Mṛdaṅga* can divide up and vibrate in 1, 2, 3, 4, or 5 parts which are separated by rectilinear nodal lines perpendicular to any chosen diameter of the membrane and give the respective overtones standing in the harmonic relation of pitch The duration of these harmonic overtones is in descending order of magnitude, being quite considerable for the first, second and third harmonics which accordingly give a fine musical effect. Tones of higher pitch than the fifth harmonic are either not excited at all in the usual manner of playing, or if excited are of too short a duration and too small in intensity to be perceptible as musical tones. In my monograph, I am giving a full discussion of the acoustical properties of the instrument together with illustrations of its mode of vibration which explain the manner in which the loading increases the duration of the tones and gives rise to the harmonic properties of the overtones. It appears in fact that the loading results in modifying the pitch of the numerous overtones which an ordinary circular drumhead is capable of giving rise to and of bringing them together in groups standing to each other in harmonic relations. The success of the arrangement depends entirely on the extent and distribution of the loading adopted and upon the arrangement provided by which the tensions of the membrane in 8 different octants may be exactly equalized. It is in fact made abundantly evident by the investigation that the acoustic properties of the instrument are not the results of mere chance but bear the evidence of the most painstaking care and skill shown in the design and construction of the instrument.

5. *The Technique of Playing the Mṛdaṅga.*

If the instrument is in itself a noteworthy piece of acoustic workmanship, still more remarkable is the manner in which its acoustic characters are utilized in actual musical practice. The drumhead is played with the hand and fingers and possesses a highly developed and finished technique. A very fair amount of practice is required even for acquiring a rudimentary knowledge of the instrument, but the finest technique can be mastered only by years of training and experience. The physical basis of the technique lies in the manner of striking the drumhead and upon the tone-quality, intensity and duration of the sounds elicited thereby. The strokes involve the exact regulation of the region of contact, the softness or hardness of the blow, its duration and force, and provide for touching the membrane with some of the fingers either during or after the blow so as to damp out certain harmonics and bring out certain others. Some of the recognised strokes provide for bringing out either the first or the second or the third harmonic practically by itself, or in combination with one or more of the five available tones. The strokes on the drumhead may be either by themselves or may be simultaneous with strokes on the base side of the drum which is tuned to one octave below the pitch of the first drumhead. Over and above this is the fact that the drumming is practically continuous and proceeds on a complex and varied metre and rythm of its own depending on the accompaniment. All this may serve to give some idea of the extraordinary degree of developement which the construction and use of percussion instruments has attained in India.

6. *Conclusion.*

The study of the Indian musical drum and of the manner in which out of the most unpromising materials has been built up a genuine musical instrument which satisfies the most stringent acoustical tests and which even now stands on a pedestal high above the types of percussion instruments known to European Music, leaves very little doubt in one's mind as to the highly-developed artistic tastes and acoustic knowledge of the ancient Hindus. The high esteem in which the instrument itself has always been held in India and the existence of many treatises in the original Sanskrit dealing elaborately with its construction and technique is not without significance. Indeed, from the references that appear in certain of these treatises, it is clear that the general nature of the acoustic results obtained with this instrument had long been known, and that the pitch and duration of the different tones obtained by striking the drumhead at different points had been fully studied. The Hindus were well aware that sounding bodies generally give rise to many different tones simultaneously, and the evidence available points irresistably to the conclusion that the development of the Indian musical drum was the result of deliberate and probably prolonged efforts to improve the tone quality of percussion-instruments by bringing the overtones into musical relation with each other. The success of the results obtained remains a striking testimony to the acoustic knowledge and skill that must have inspired those efforts.

ON TAUTOMERIC CHANGES IN THE PHENYLHYDRAZONES OF ORTHO-ALDEHYDIC AND 1-4 ALDEHYDIC ACIDS:

Profulla Chandra Mitter, M.A., Ph.D., *Sir Rashbehari Ghosh Professor of Chemistry,*

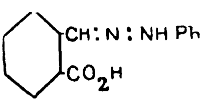
and

Judhisthir Chandra Das, M.Sc.

It was shown by one of us[*] that aromatic ortho-aldehydic acids like opianic and nitro-opianic acid react in ethereal solution with free phenylhydrazine base with the formation of phenyl hydrazones which subsequently tautomerise to phenylhydrazo-phthalides. On oxidation with mercuric oxide in acetone solution, the phenylhydrazo-phthalides are converted into phenyl-azo-phthalides. It was subsequently shown[†] that on adding a solution of phenylhydrazine hydrochloride to a nearly neutral solution of an aromatic orthoaldehydic acid like phthalaldehydic acid, phenylhydrazo-phthalide is obtained which can be oxidized to phenyl-azo-phthalide. The oxidation is best performed with mercury acetamide in acetone solution.

Treatment with acetic acid converts the phenylhydrazo-phthalide into a phenyl-phthalazone in each case.

Thus :



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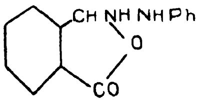

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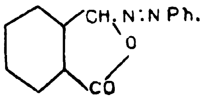

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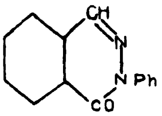

IV.

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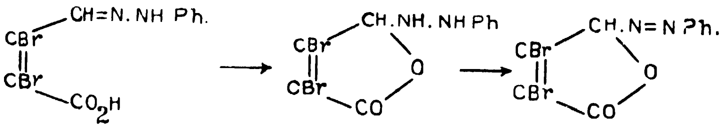
[*] Mitter and Sen, *J.C.S.*, *111*, 988 (1917).

[†] Mitter and Sen, *J.C.S.*, *115*, 1145 (1919)

The similarity in the constitutions of aromatic ortho-aldehydic acids and 1 : 4 aldehydocarboxylic acids in the aliphatic series led us to investigate the action of phenylhydrazine on mucobromic and phenoxy-mucobromic acids and to try the action of mercuric acetamide on the hydrazo-derivatives that are formed in the first instance. As was anticipated, phenyl-hydrazo-furfuranes were formed which on oxidation gave phenylazo-furfuranes. On treatment with glacial acetic acid on the other hand, the phenyl-hydrazo-derivatives gave rise to pyridazones.



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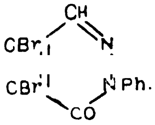
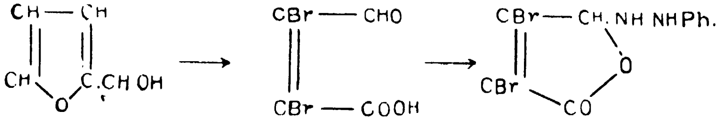


(VIII)

The action of phenylhydrazine on mucobromic and phenoxy-mucobromic acids was studied by Bistrzycki and Simonis * and later by Bistrzycki and Herbert † who obtained mucobromic acid phenylhydrazone and phenoxy-mucobromic acid phenylhydrazone identical with α-Keto—α phenylhydrazo -ββ′ dibrom—αα′ dihydro furfurane, and α—Keto—α′ phenylhydrazo—β-phenoxy—β′ brom—α-α′ dihydro furfurane described by us.

The starting material for the preparation of mucobromic and phenoxy-mucobromic acid is either furfurol or pyromucic acid, and it is interesting to note that the furfurane ring which is opened up by bromine is closed again with the help of phenylhydrazine.



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* Ber., 32, 535 (1899).

† Ber., 34, 1012 (1901).

EXPERIMENTAL.

Interaction of mucobromic acid and phenylhydrazine.

Mucobromic acid is very soluble in ether. One grm. of mucobromic acid was dissolved in 2 c.c. of ether and the ethereal solution of phenylhydrazine liberated from ·5 gr. of phenylhydrazine hydrochloride by sodium hydroxide was added to it with ice cooling. It was allowed to stand for half an hour. On evaporation of the ether, a reddish yellow mass was left which on crystallisation from slightly warm alcohol was found to melt at 102°-103°C

As the phenylhydrazo-compound is very soluble in ether, it was prepared by the following method.* 1 grm. of mucobromic acid was dissolved in 5 c.c. of water and to it ·2 grm. of crystallised sodium carbonate was added. This solution was cooled in ice bath and to it a clear solution of ·5 grm. of phenylhydrazine hydrochloride was added. At first, the solution became greenish yellow, and then a yellow flocky mass came down. It was then filtered off, and washed with cold water and dried on a porous plate. It was found to melt at 102°-103°C. It is easily soluble in dilute sodium carbonate, sodium hydroxide and also in sodium bicarbonate solution. It goes into solution in glacial acetic acid, alcohol and ether, etc. Nitric acid and hydrochloric acid produce no coloration. Even strong sulphuric acid does not give any colour with the freshly prepared substance.

0·1528 grm. gave 10·2 c.c. N at 26°C. and 760 m.m.

$C_{10}H_8O_2N_2Br_2$.

Calc., N=8·06.

Found, N=7·53.

Oxidation of the Compound.

One grm. of the phenylhydrazine derivative was dissolved in 10 c.c. dry acetone and 0·9 grm. of mercury acetamide was added to it. The mixture was heated under reflux on a water bath for 6 hours. It was filtered and the filtrate was kept in a vacuum desiccator for evaporation of the acetone. The residue was washed with dilute soda solution and then with water and crystallised from acetone. The yield was 0·2 grm.

The substance melts with decomposition at 136°C. It dissolves readily in acetone, ether, alcohol, chloroform and benzene. The colour of the substance is brownish yellow.

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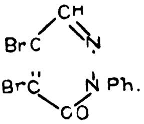
* *Ber.*, *34*, 1013 (1901).

It gives all the reactions of an azo-compound. With concentrated sulphuric acid it gives a blue-violet coloration. An alcoholic solution of the substance gives a pink colour on the addition of a drop of alkali solution.

0·1119 grm. gave 0·1413 CO_2; 0·0163 H_2O;
0·1146 gave 8·2 c.c. of N at 26°C. and 760 m.m.
$C_{10}H_6O_2N_2Br_2$ (Formula 7).
Calc., C=34·68; H=1·73; N=8·09.
Found, C=34·44; H=1·62; N=8·07.

Preparation of 1-phenyl 4·5-dibromo pyridazone.



0·5 grm. of the hydrazo compound was dissolved in 15 c.c. of glacial acetic acid, and the solution thus obtained was heated to boiling for a few minutes. To the hot solution, hot water was added, till it became turbid. On cooling white flaky crystals came out and were found to melt at 144°C. It is insoluble in sodium carbonate and sodium hydroxide and does not give any coloration with strong sulphuric acid. It is the ring compound of Bistrzycki and Simonis.*

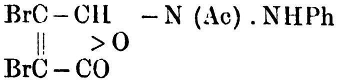
It is insoluble in sodium carbonate and sodium hydroxide, but soluble in alcohol, glacial acetic acid and ether.

0·0710 grm. of the substance gave at 26°C. and 760 m.m. 0·5 c.c. of nitrogen.

$C_{10}H_6O\ N_2Br_2$
Calc., N=8·50.
Found, N=7·95.

Action of acetic anhydride upon the hydrazo compound.

Formation of



$$\begin{array}{l} BrC-CH\quad -N(Ac).NHPh \\ \quad \| \qquad >O \\ BrC-CO \end{array}$$

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* *Ber.*, 32, 557 (1899).

One grm. of hydrazo compound was treated with a slight excess of acetic anhydride and this was allowed to stand over-night. Some reddish brown crystals were found to form. To this water was added and the crystals were filtered, washed with water and was treated with alcohol. Golden yellow crystals insoluble in alcohol were obtained. It melts at 140 to 141°C. It is insoluble in sodium carbonate and sodium hydroxide. A little of the substance was taken and evaporated to dryness with sodium hydroxide on the water bath. The whole of the salt was heated with arsenious oxide in a test tube and was found to give the smell of cacodyl.

0·1106 grm. gave 0·1515 grm CO_2 0·0242 grm. H_2O

$C_{12}H_{10}O_3N_2Br_2$.

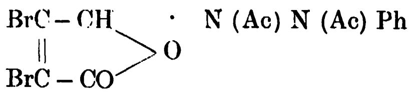
Calc., $C=36·93$; $H=2·57$.

Found, $C=37·53$; $H=2·42$.

Action of Acetyl chloride upon the phenylhydrazine derivative.

One grm. of the phenylhydrazine derivative was dissolved in ether. It was added to an ethereal solution of ·3 grm. of acetyl chloride, and kept at the ordinary temperature for an hour and a half in a desiccator over sulphuric acid. HCl gas was found to evolve. Then the whole was heated on a water bath to drive off the ether and acetyl chloride. A white crystalline mass was left. It was dissolved in boiling methyl alcohol and on spontaneous evaporation, crystals began to come out. It was then filtered and the residue dried on a porous plate. It melts at 138·5 to 139°C.

It is insoluble in sodium hydroxide solution, hot or cold. A little of the substance was boiled with sodium hydroxide, and then the whole mass was evaporated to dryness. On heating with arsenious oxide the smell of cacodyl was perceptible. The analysis of the substance showed that it is the diacetyl derivative.



0·1126 grm. of the substance gave 0·1609 grm. of CO_2; 0·0332 grm. of H_2O

$C_{14}H_{12}O_4N_2Br_2$.

Calc., $C=38·89$; $H=2·82$.

Found, $C=38·97$; $H=3·28$.

Action of Phenylhydrazine upon Phenoxy-mucobromic acid.

Two grms. of the sodium salt of phenoxy-mucobromic acid was dissolved in water. This solution was cooled in ice bath, and a clear solution of 2 grms. of phenylhydrazine hydrochloride was added rapidly. At first a greenish yellow solution appeared and then a light yellow crystalline mass came down on stiring. This was crystallised from hot alcohol and was found to melt at 119·5 to 120°C.

It is soluble in sodium carbonate solution, sodium hydroxide, and in alcohol, acetone and glacial acetic acid from which it gives the pyridazone derivative. Sulphuric acid produces a red colouration.

0·1192 grm. of the substance gave 0·2332 grm. of CO_2, ·0355 grm. of H_2O.

$C_{18}H_{13}O_3N_2Br$.

Calc., C=53·35; H=3·31.

Found, C=53·18; H=3·60.

Oxidation of the phenylhydrazine derivative.

One grm. of the phenylhydrazine derivative was dissolved in 25 c.c of acetone and 1·3 grm. mercury acetamide was added to the solution. Then the whole was heated under reflux by steam from a water bath for six hours and kept at ordinary temperature over night. It was then filtered and the acetone evaporated A crystalline mass was obtained. It was washed with hot water twice and then treated with dilute sodium carbonate solution for half an hour to free it from any unchanged hydrazo compound. Afterwards it was washed several times with water and then recrystallised from dilute acetone.

The substance melts at 158° to 159·5°C.

It is an orange yellow crystalline substance, soluble in acetone but insoluble in sodium carbonate. It gives with concentrated sulphuric acid a greenish blue colour.

To the alcoholic solution of the substance was added a dro, of sodium hydroxide solution. A pink colouration was developed disappearing just on the addition of more alkali.

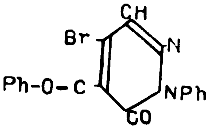
0·1196 grm. of the substance gave 0·2338 grm. of CO_2 0·0361, grm. of H_2O.

0·1257 grm. of the substance gave 9 c.c. of N_2 at 26·5°C and 760 m.m..

$C_{18}H_{11}N_2O_3Br$.

Calc., C=53·48; H=3·06; N=7·80.
Found, C=53·31; H=3·36; N=8·08.

Preparation of 1-phenyl 4-bromo 5-phenoxy-pyridazone.



0·5 grm. of the phenylhydrazine derivative was dissolved in glacial acetic acid and heated to boiling for a few minutes. Then on diluting with boiling water till the solution became turbid and afterwards on cooling a white crystalline mass appeared. It was found to melt at 113° to 114°C.

It is insoluble in sodium carbonate or even in sodium hydroxyde. It is identical with the ring compound obtained by Bistrzycki and Herbert.*

0·1900 grm. of the substance gave 12·4 c c. of nitrogen at 24°C and 760 m.m.

$C_{16}H_{11}N_2O_2Br$.
Calc., N=8·16.
Found, N=7·42.

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* *Ber.*, *34*, 1013.

PASPALUM, DIGITARIA AND ANASTROPHUS: A STUDY.

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The present paper is the result of a detailed examination of the collection of the material of the genus Paspalum, incl. Digitaria and Anastrophus, contained in the Herbarium of the Royal Botanic Gardens, Sibpur, supplemented to a limited extent by observations in the field. The investigation referred to is part of the work undertaken by the writer with a view to the publication of a "Flora of North-Eastern India" of the type of Theodore Cooke's Flora of Bombay. The constant use to which Cooke's Flora is put by botanists on this side of India proves the desirability and urgency of bringing out a similar publication dealing with the floras of Bengal, Sikkim, and Assam. Teachers of botany in the colleges scattered all over the country find it impossible to consult at frequent intervals larger collections such as those housed in the Herbarium of the Royal Botanic Gardens at Sibpur; moreover, the whole of the flora of this part of the country requires to be worked over in detail, before it is possible to compose "pocket floras," which would enable field botanists and in general persons who take an interest in botanical studies to identify satisfactorily and without undue loss of time any phanerogamic plant they may meet in forest or field.

Many of the species of Graminaceae, being cosmopolitan, form excellent subjects for the study of variations; on the other hand, their variability and polymorphism, the great dependence of the morphological characters on ecological conditions, render the establishment of well-defined species, sub-species and varieties, or even genera, a matter of considerable and sometimes nearly unsurmountable difficulty. A great deal of work remains yet to be done in that direction by work in the field and growth experiments carried out in different parts of this country. What is sometimes declared to be "rich" herbarium material turns out on closer examination to be very deficient, a fact which is not astonishing considering the vast extent of the area Indian botanists have to deal with; and in the study of grasses this area has to be extended far beyond the limits of India.

As the Flora of British India is practically the only work which botanists on this side of India are able to use when confronted with the task of identifying specimens of phanerogams, the terminology adopted in this paper will be chiefly that used in the monumental work just referred to. We shall, therefore, usually use the combination, say, *Paspalum sanguinale,* in preference to either *Digitaria sanguinalis* or *Panicum sanguinale.* After all, Paspalum, Digitaria, Anastrophus and Panicum are closely related form-circles; and although glume I of Panicum is often absent in grasses belonging to the Digitaria group and nearly always absent in the Paspalum form-circle, in certain species of Digitaria the presence or absence of glume I appears to follow no definite rule, and in *Paspalum protensum*, Trin., a native of Brazil, which in every other respect is a typical Paspalum, glume I is as well, or even better, developed as in many species of Digitaria. (See fig. 1, pl I). The statement "Lower involucral glume absent" cannot, therefore, be applied to *Paspalum protensum,* although it is applicable to all Indian species of that genus, taken the latter in its restricted sense.

A character which in many cases permits the easy discrimination of Paspalum and Digitaria is the shape of the spikelet. The spikelets are, in a great number of species of Paspalum, orbicular or broadly elliptic, whilst in Digitaria they are ovate-or elliptic-oblong to ovate- or elliptic-lanceolate. This difference, however, is not so well marked in those species of Digitaria the length of whose spikelets is two millimeters or less, whilst it is quite evident in species whose spikelets measure 2·5 to 3·5 millimeters in length.

The character which for the purposes of discriminating Digitaria from Paspalum is more reliable than any other is the nervation of glume III, the abaxial or dorsal flowering glume. In typical species of Paspalum, such as *Paspalum scrobiculatum*, Linn., *P. conjugatum*, Berg., *P. compactum*, Roth, and most of the American species, the intermediate nerves, i.e. the nerves next to the midnerve, are remote from the latter and close to the line of inflection of the marginal parts of the glume, leaving comparatively large areas on either side of the midnerve free from nervation. As a necessary consequence the intermediate as well as the lateral nerves exhibit a correspondingly marked degree of curvature. In Digitaria, on the contrary, the nerves of glume III are more equally spaced, and the intermediate nerves are often quite or nearly parallel to the midnerve. (See figs. 2 and 3, pl. I.) There are cases, however, in which these distinctions break down to a lesser or greater extent. Cases in point are the following Indian examples :—

(*a*) PASPALUM DISTICHUM, Linn.

This species is one of the most aberrant of the Indian species of Paspalum in the narrower sense of the term, and the following statements regarding form and nervation of glume II (the adaxial involucral glume) and glume III (the abaxial flowering glume) may prove of interest. The midnerve of glume II may or may not be developed; it may reach the tip of the glume or stop short somewhere half-way up; it may be absent in all the spikelets of a specimen, or it may be developed in the uppermost spikelets and absent in the lowermost ones, or *vice versa*. The first case, that in which glume II has no midnerve, has been observed by the writer in specimens from Formosa, Borneo, and various places in India; specimens in which the midnerve is developed in the uppermost spikelets and absent in the lowermost hail from Singapore and the Sunderbans; specimens with glume II traversed by a midnerve in the lower spikelets and not in the uppermost ones are from Pahang, whilst in specimens from Karnal in the Panjab the midnerve of glume II is strongly developed in both upper and lower spikelets. The length of glume II varies usually between 3 and 3·5 mm., but may not exceed 2·7 mm.; when flattened out, its breadth is seen to vary between 2 and 1·8 (sometimes 1·5) mm. A character which appears to be tolerably, if not quite constant consists in a tiny tuft of hairs at the tip of glume IV, the upper chartaceous or coriaceous flowering glume. (See figs. 4 *a-d*, pl, I).

(*b*) PASPALUM LONGIFLORUM, Retz.

In herbaria, specimens of this species are often found mixed up with *Paspalum distichum*, Linn., *P. Royleanum*, Nees, *P. sanguinale*, Lamk., and *P. pedicellare*, Trin., but particularly with *P. Royleanum*. From all these species it can at once be distinguished, and that absolutely, by the nature of the hairs on glumes II and III, a character which, according to Sir Joseph Hooker, was first pointed out to him by Dr. Stapf (see Flora of British India, Vol. VII, p. 19 under *P.*

Royleanum and also p. 18 under *P. longiflorum*). These hairs are stated in the F.B.I. to be "very slender and as it were crisp or wrinkled." They may also be described as knarled or crinkled. Sometimes they are straight at the tip, but very commonly they are hooked after the fashion of a bishop's crook. (See fig. 5 *a–f*.) The fact is certainly remarkable that a character of this description, the usefulness of which to the plant is far from evident, should be so absolutely constant over an immense area. Glume I is often, but not at all always, absent. The length of the spikelets varies from 1·3 to 2·0 mm.

(*c*) PASPALUM ROYLEANUM, Nees.

Although, in collections, specimens of *P. Royleanum* and *P. longiflorum* are mixed up to a considerable extent, the nature of the hairs of glumes II and III allows of their easy and certain discrimination, and the question whether a specimen belongs to either one or the other species—other alternatives being supposed to be excluded—can be settled at once by examining either glume II or glume III under one of the medium powers of a compound microscope. Fig. 6 shows such hairs from glume II of specimens of *P. Royleanum* from different localities. It will be noticed that the hairs exhibit a considerable amount of variation in length and form. They are always gland-tipped, but the glandular part may be obovoid with the upper end rounded or depressed, or it may be distinctly spindle-shaped; the shank of the glanduliferous hair may be comparatively short and the gland may be even subsessile, or the shank may be slender and considerably longer than the glandular head. These different descriptions of hairs may occur side by side of each other. (See fig. 6.) The crown of stiff hairs on the pedicels of most forms of *P. Royleanum* is another character which in the majority of cases allows of the ready discrimination of the species under review from *P. longiflorum*. (See fig. *a–d*, pl. I) It is, however, not as reliable as the nature of the hairs on glumes II and III, as already indicated by Sir Joseph Hooker in the Flora of British India. It is stated there that in African specimens those hairs are longer than the spikelets; this is, however, never the case in Indian specimens; indeed, as also mentioned by Hooker, in certain Ceylon specimens "the pedicels are hardly setulose." It may occur that a casual observer may be led astray with regard to this point by herbarium specimens which are doubtlessly specimens of *P. longiflorum* or *P. pedicellare* being erroneously named *Paspalum Royleanum.*

On the other hand, in certain specimens from the Khasia Hills, Pegu, and Singapore which certainly are forms of *P. Royleanum* the crown of setulose hairs is entirely absent, whilst in other specimens it may be represented by a single longish bristle, the other hairs being of a minute size. In one of these specimens, collected by Ridley near Singapore, otherwise hardly separable from *P. Royleanum*, not only are the pedicels very scantily scabrid and only with a trace of a crown of setules, but also glumes II and III are entirely glabrous, without a sign of gland-tipped hairs. As regards glume II, it may be stated that, in general, it is very short or nearly obsolete in specimens from Rajputana, Central and Southern India. In these specimens the hairs of glumes II and III are comparatively short and tipped with short-obovoid glands; in certain specimens from other areas glume II is more developed and may even nearly equal glume III in length. The spikelets of *P. Royleanum* are inserted in clusters of three or two, more rarely four, alternately on either side of the dorsal ridge of the rhachis; the lower pedicel is about half a millimeter in length, the middle one is about twice as long and the uppermost of the three is three or more times the length of the lowest one. In a very large number of specimens the length of the spikelets varies between 1·1 and 1·7 mm. Certain specimens, however, named *P. Royleanum*, and certainly closely related to that species, have spikelets 2·1 to 2·5 mm. in length; in all these specimens the pedicels are crowned with a tuft of bristle-hairs; the hairs on glumes II and III are slender and pass gradually into an oblong-obovoid gland-like expansion bearing a minute conical tip; glume II is well developed, 1·6 to 2·0 mm. long, narrow-oblong and three-nerved. The distribution is rather remarkable: it occurs in Yunnan (Henry's Collection), in the Myrung Hills, at Kotagiri in the Nilgiris (Collection Gamble) and near Sahebganj in the Rajmahal Hills. In the latter specimen glume I is distinct and about 0·1 mm. long. Fig. 8 *a–e* represents this form, which, having been first noticed by the writer on sheets of Henry's Yunnan collection, he proposes to call *var. yunnanensis*. These forms are possibly mutations which have independently originated at different centres.

(*d*) PASPALUM TERNATUM (Hochstetter), Hook. fil. (See figs. 11 *a–e* pl. I.)

This species seems to be even more closely related to *P. Royleanum*, than to *P. ambiguum*, particularly if the form distinguished by the writer as *P. Royleanum var. yunnanensis* should ultimately prove to be specifically inseparable from *P. Royleanum.* The pedicels are beset with stiff hairs, which gather into a crown of setulae at the apex; the hairs of glumes II and III gradually widen out into a clavate tip, and the spikelets vary in length between 2 and 2·5 millimeters. The basal spikelets commonly occur in clusters of three with the pedicels 0·5, 1·0 and 1·5, or 0·5, 2·0 and 3·0 mm. in length. The wings of the rhachis are as broad as, or narrower than the midrib.

(*e*) PASPALUM PEDICELLARE, Trinius.

There is usually no great difficulty in discriminating forms belonging to this species from forms belonging to *P. Royleanum*, the length of the spikelet fluctuating on either side of 1·5 mm. within very narrow limits. The crown of setulae on the apex of the pedicels of most forms of *P. Royleanum* appears to be never developed in *P. pedicellare*. The glandular expansion of the hairs of glumes II and III is oblong-obovoid or oblong-ellipsoidal, never depressed-ellipsoidal, and the shank of the hairs is always considerably longer than the glandular tip. The upper spikelets are always geminate, and the lower ones occur in clusters of more than two A peculiarity of these clusters consists in that the pedicels of the single spikelets arise at different levels, as will be seen from figs. 9, *a-f* which are from the district of Manbhum; in the specimen referred to the inflorescence consisted of twenty-one clusters. In a number of measured specimens from different localities the wings of the pedicels had a width less than half the width of the midrib.

(*f*) PASPALUM JUBATUM, Griesbach.

As the description of this species as given in the Flora of British India is rather scanty, the species is here described somewhat more fully, but the advent of more plentiful material of this seemingly rather rare species may cause slight modifications in the final description. Rootstock short. Rootlets wiry, issuing from the rootstock and the lowermost internodes of the culm Culm single, erect, with the inflorescence 60 to 130 cm. in height, near the base about 3 mm in thickness, smooth and shining. Internodes 6-15 cm. long. Nodes constricted, short, brown, glabrous. Leaves glabrous. Sheaths as long as, or somewhat shorter than, their internodes, with well-marked filiform ribs. Ligule short, rounded. Blade linear, running out into a fine point, 15–40 cm. long, 3–6 mm. wide, with a well-marked stout-filiform midnerve and 3–4 somewhat thinner side-nerves and thin intermediates. Peduncle about 2 mm. thick at its base, 25 to above 30 cm. in length. Main rhachis 4- or 6-angular in cross-section, ending in a terminal raceme and giving off at intervals of 15 to 5 mm. five to seven lateral suberect racemes of various lengths, the longest about 20 cm., the basal ones being sometimes no more than 1 cm. The rhachis of the racemes ± sinuous, about 0·3 mm. in width with a dorsally flattish, ventrally sharp midrib and very narrow, minutely scabrid wings. Spikelets in clusters of five to two, mostly arising at about the same level, arranged alternately along the midrib of the rhachis, 1·6–1·8 mm. long, 0·6–0·8 mm. in width, lanceolate-oblong, very sharply acuminate, pale-coloured or dark-purple; pedicels of lowest spikelets about 0·5 mm. long, of the higher ones ± sinuous and increasing to 3-4 mm. in length Glume I absent. Glume II thin-membranous, convex, elliptic, 3-nerved, dorsally minutely and softly pubescent. Glume IV thin-membranous, flat, 5-nerved, dorsally minutely pubescent, intermediate ones straight and parallel. Hairs of glumes II and III of unequal lengths, usually gradually widened into a slender obovoid head. Glume III cartilaginous, lanceolate-oblong, acute, dorsally convex, striolate, brown, about 1·5 mm. long, margins incurved, flaps paler-coloured, their edges meeting. Palea of glume IV 1·2 mm. long ovate, acute, back chartaceous, margins incurved, thinner, gaping. The material available is not sufficient to give a detailed description of stamens, pistil and grain. For cluster of spikelets and hairs from glumes II and III of *Paspalum jubatum* see figs. 10 *a–b*.

(*g*) PASPALUM SANGUINALE, Lamk. (= *Digitaria sanguinalis*, Scopoli, = *Panicum sanguinale*, Linn.).

Although the nine varieties distinguished in the Flora of British India are connected by intermediate forms, some of which may ultimately prove to be hybrids, it is nevertheless possible to separate certain of the Indian form-circles and raise them to the dignity of subspecies or species of the second order. After a detailed and protracted study of the available herbarium material as well as numerous fresh specimens the writer has arrived at the conclusion that the following form-circles can be distinguished from each other with tolerable ease: (1) subsp. *P. cruciatum* (Nees); (2) subsp. *P. commutatum* (Nees); (3) subsp. *P. extensum* (Nees); (4) *P. ciliare* (Retz.); (5) *P. corymbosum* (Roxb.); (6) *P. pabulare* (Aitch. and Hemsl.).

(1) PASPALUM CRUCIATUM.

It is, as a rule, quite easy to distinguish this subspecies from *P. commutatum* and other subspecies by the ovoid or oblong-ovoid, subabruptly and shortly acuminate spikelets and more particularly by the shape of glume II, which when flattened out is seen to be broadly ovate, rather obtuse, 3-nerved and 1–1·5 mm. long. (See figs. 12 *a-f* pl. II.) The grain is rather squat, and its shape may have first induced the Khasias to take it in cultivation. The writer has little doubt about the plant cultivated in the Khasia Hills being derived from *P. cruciatum* and not from *P. commutatum*, although with regard to the form cultivated in Sylhet its derivation from *P. cruciatum* is more doubtful. The cultivated form is stouter and taller than the forms growing wild all along the Himalaya from Gilgit to Bhutan extending into the Assam Hills and probably farther east. Besides the forma *culta* and the forma *typica* we may notice a form with hirsute leaf-sheaths from Lachung (forma *lachungense*) and a form from the Khasia Hills (forma *setulosa*) which has the squat shape and the subabruptly cuspidate glume IV of forma *typica*, but in which glume II is oblong and glume III bears a row of bristle-hairs along the intermediate nerves and has a densely ciliate margin, thus being to a certain extent intermediate between subsp. *P. cruciatum* and subsp. *P. ciliare*. (See figs. 13 *a-c*.)

(2) and (3) Subsp. PASPALUM COMMUTATUM (= *Digitaria commutata*) and Subsp. PASPALUM EXTENSUM, Nees. (See figs. 14 and 15.)

It is sometimes quite impossible to decide whether a certain specimen should be assigned to *Paspalum commutatum* or to *Paspalum extensum*, but as in other cases the discrimination can be effected with comparative ease, the writer proposes to keep the two form-circles apart, at least for the present, basing the distinction on the adaxial involucral glume (glume II), which in subsp. *P. commutatum* is 1·8 to 2·8 mm. long, and in most cases ½ to ¾ (rarely only ⅓) the length of glume IV, whilst in subsp. *P. extensum* glume II is usually 0·6 to 1·2 mm. long and less than ½ the length of glume IV, being in rarer instances nearly obsolete. A study of the variation in the absolute and relative length in otherwise typical specimens of subsp. *P ciliare* has convinced the writer that the separation of *P. commutatum* and *P. extensum* as subspecies is really artificial and can be defended only on grounds of expediency. In this connection arises also the question of the position of var. *Rottleri* and var. *debile* of the Flora of British India. As regards *Digitaria debilis* (see figs. 16 *a*, *b*, pl. II) which in its typical form hails from the Mediterranean Region, we may accept Parlatore's definition: "Digitaria spicis subdigitis, filiformibus, subquinis, spiculis oblongo-lanceolatis, gluma inferiore nulla, superiore *flosculos superante* paleaque flosculi neutri cuspitatis, subseptemnervibus, puberulis, vaginis foliorum inferiorum villosis." Parlatore adds: "Questa specie per mancanza della gluma inferiore e per lo svoluppo maggiore della superiore avvicina la Digitaria ad Paspalum." We may therefore assume that the character distinguishing the true *Digitaria debilis* from allied forms of *Paspalum sanguinale* lies in the adaxial involucral glume exceeding in length the flowering glumes. J. W. Bews also in his interesting treatise on "The grasses and grass lands of South Africa" (1918) distinguishes *Digitaria debilis* from *D. sanguinalis* by the former having the upper glume (our glume II) long-acuminate, exceeding the upper valve (our glume IV), and the latter (*D. sanguinalis*) having the upper glume shorter than the upper valve. In the numerous specimens from Ceylon, the Andamans, Nicobars, Bengal, Burma, the Malay Peninsula, Java and Celebes in the Sibpur Herbarium and named var. debilis, in many instances on the authority of Sir J D. Hooker, the length of glume II varies between 1 and 2 mm., whilst that of glume IV varies between 2·2 and nearly 3 mm., that is to say, glume II is always shorter than glume IV. The writer is, therefore, of opinion that *var. debilis* of the Flora of British India is not identical with the *Paspalum debile* of Poiret or the *Digitaria debilis* of Parlatore. The Eastern form is evidently, as already hinted in the Flora of British India, nothing but a soil-variety, in most cases probably of *P. commutatum*, in other cases of *P. ciliare*. As regards var. *pruriens*, the F.B.I. states that glume I (our gl. II) is nearly as long as glume III (our glume IV). The writer is not able to confirm this statement, as in all the specimens marked *var. pruriens* in the Herbarium of the Sibpur Botanical Gardens glume II is distinctly shorter than glume IV, the specimens, as a matter of fact, differing in no respect from what we regard as subsp. *P. extensum*. Miquel also in his Flora Indiae Batavae says of *P. pruriens* that "gluma superior parva spiculae $\frac{1}{4}$ aequans," whilst in his *var. Arnottiana* glume II is said to reach half the length of the spikelet. *Paspalum pruriens* and *Paspalum extensum* have therefore to be merged into a single subspecies; *P. pruriens* cannot even be considered a separate variety and has to disappear from Indian "floras."

var. Rottleri, which is stated in the F.B.I. to be a dwarf form of *var. commutata*, is related to *P. extensum* as var. "debile" is related to *P. commutatum.* (See figs. 17 *a—c.*) In other words *var. Rottleri* and var. "debile", are soil-varieties of subsp. *P. extensum* and Subsp. *P. commutatum* respectively. In this view we are confirmed by a study of the soil-forms of subsp. *P. ciliare*, the otherwise typical forms of which exhibit a similar variability when growing in different localities or on different soils in the same locality. The character relied on in the F.B.I. as distinctive of *var. Rottleri*, namely that the rhachis of the spike is "stouter broader green, the wings two or three times broader than the midrib" is not constant, and the wings may be only 1 to $1\frac{1}{2}$ times as broad as the midrib. As a matter of fact, *var. Rottleri* is not at all as common as suggested in the F.B.I. It is best to restrict the name to the soil-forms in which glume II is thin-membranous, ovate or ovate-oblong, usually finely 3-nerved, sometimes faintly 1-nerved or even nerveless, commonly 0·8-1·1 mm. long and about half as broad, rarely glabrous, usually ciliate with the hairs sometimes as much as 1 mm long. The name *var. pseudodebilis* may be assigned to the small soil-form of *P. commutatum*, the *var. debile* of the F. B. I. in which glume II is ovate- or triangular-lanceolate, sometimes ovate-oblong, subacute, 1–2 mm. long, ($\frac{1}{3}$)-$\frac{1}{4}$-$\frac{1}{5}$ as broad, 3-nerved with the lateral nerves, as a rule, gradually converging from base to apex, margin adpressedly or villously ciliate, back usually pubescent between median and lateral nerves

The following are the localities at which the specimens examined by the writer have been gathered; the subspecies and varieties are taken in the sense indicated above.

P. commutatum typicum.–

Kurum Valley (Aitchison), Lahoul (Stoliczka), Bashahr (Lace), Chamba (Lace), Busahir (Brandis), Pangi (Lace), Simla (Gamble), Mussoorie and Dehra Dun (King), Almorah (Strachey and Winterbottom), Nepal (Wall. Cat. 8681 J), Sikkim (J. D. Hooker, C. B. Clarke), Assam (Jenkins), Khasia (Mann), Silhet (Wall. Cat. 8681), Burma (Wall.

Cat. 8683, Parish, Kurz), Coimbatore (B. Schmid), Kodai Kanal (Saulières), Naduvatam 6,000′ (Bourne), Chota Nagpur (Wood).

P commutatum, var. *pseudodebilis* :—

Bengal (various collectors), Burma (Kurz), Great Coco (Prain), Malay Peninsula (various collectors), Andamans and Nicobars (Kurz), Ceylon (Thwaites), Java, Celebes

P. extensum typicum :—

Bihar (Coll.), Rajmahal Hills (Kurz); Bengal (Coll.), Hill Tippera (Debbarman), Cachar (C. B. Clarke), Assam (Mann, Coll.), Jaintia Hills (Mann); Burma (Kurz), Malay Peninsula (Coll.), Laccadive Islands (Investigator). Java, Borneo, S. India (Wight, Bourne, Saulières) Cochin (Meebold).

P. extensum var. *Rottleri* :—

Sikkim (Kurz, C. B. Clarke), Bihar (Coll.), Bengal (Coll.), Burma (Kurz), Great Coco (Prain), S. India (Heyne).

(4) Subsp. PASPALUM CILIARE. (See figs. 18 *a*, *b*.)

The results of a detailed investigation into the form-circle of *P. ciliare* will not be ready for publication till the end of the rainy season of 1921, as these investigations involve growth experiments by which alone a number of problems can be elucidated and doubtful points cleared up. For the present the writer is constrained to confine himself to the following remarks :—

The best distinguishing character of this form-circle is the indumentum of glume III (the abaxial flowering glume) of the pedicelled spikelet as it appears at a later stage, especially in fruiting specimens. The marginal strips are beset with a dense fringe of rather soft hairs, the upper part of which is incurved and in the dry state interwoven into a kind of rim, which on cursory examination may be mistaken for a marginal nerve and which is commonly strengthened by bristle-hairs. The intermediate nerves are strongly developed, and disposed along them is a similar dense fringe of upward-turned softish hairs intermixed with bristle-like hairs. The bristle-hairs may reach a length of 1.5 mm., but are usually 1 mm. or slightly less in length. In the sessile or subsessile spikelets a well-developed fringe of hairs along the intermediate nerves as well as bristle-hairs are not uncommonly entirely absent. It does happen that, as the grains of the pedicelled spikelets ripen sometime before those of the sessile ones and fall off before the latter are nearly mature, the specimens are often erroneously ascribed to *P. commutatum* or *P. extensum*, or even *P. Rottleri* or *P. debile*. On careful scrutiny of the sheets it is often possible to spot some stray stalked spikelets, when the true relationship of the specimen stands revealed. In other cases it may be practically impossible to assign specimens to their proper place, all the pedicelled spikelets having been lost. Unfortunately, the bristle-hairs ,which are so characteristic of many forms of subsp. *P. ciliare* occur in rare instances in other subspecies, such as *P. cruciatum.* The specimens from the Nicobars referred to in the F.B.I. as belonging to *var. debile* are evidently soil-forms of the true subsp. *P. ciliare.* The statement that "var, ciliare" has few (2–6) spikes in an inflorescence is generally correct, but the writer has met vigorous specimens of undoubted *P. ciliare* with as many as fourteen spikes. (Compare fig. 17).

(5) Subsp. PASPALUM PABULARE (= *Panicum pabulare* of Aitchison and Hemsley) is sufficiently distinct to be raised to the dignity of a subspecies. Nothing need be added here to the description given by Aitchison and Hemsley in the Journal of the Linnean Society, Vol. XIX, p. 190. (See figs. 19 *a—e*)

(6) Subsp. PASPALUM CORYMBOSUM (or *Digitaria corymbosa*). (See figs. 20.)

The writer has little doubt about this form, which is referred to in the Flora of British India as *var. Griffithii,* being identical with Roxburgh's *Panicum corymbosum*, which is doubtfully referred to *Paspalum sanguinale* in the F.B.I. whatever the name bestowed on this form-circle, it is certainly furthest removed from the typical *Paspalum sanguinale* and when a more plentiful material will be available will probably have to be raised to the dignity of a species of the first order.

The following is a full description of the subspecies as known to the writer.

Rootstock stout, short. Central branches erect, lateral ones often prostrate and rooting at the nodes, finally ascending. Culms terete, solid, 30–120 cm. (and more) in length, 2–4 mm. thick near the base pale-coloured, smooth. Nodes constricted. Internodes (the middle ones) 4–12 cm. long, lower ones shorter. Leaves: Sheaths as long as or shorter than their internodes, ribbed, the ribs either equally strong or one stouter rib alternating with three finer ones, glabrous ± densely hispid with bulb-based hairs; ligule conspicuous, transversely oblong, somewhat rounded and erose along the upper margin, 2 3 mm. long; blade lanceolate-linear, rounded at the base or sometimes attenuated into a distinct petiole, which may be as much as 2·5 cm. long: blade attenuated into an acute apex, of intermediate leaves 8–25 cm. long, 12–25 times its greatest width, glabrous or ± densely beset on both surfaces with bulb-based straight slender hairs; midrib sharply defined on both surfaces; right and left margins alternately undulate, both margins and surfaces scabrid. Peduncle slender, glabrous, 20–50 cm. long. Inflorescence corymbose, consisting of 6 to 15 simple or compound racemes, the lowest ones verticillate, the upper subopposite and alternate. Main rhachis trigonous or rectangular in cross-section Racemes slender, strict or flaccid, divergent or drooping, longest 10–15 cm., some of the basal ones often only 1–2 cm. long. Spikelets in pairs, one very shortly the other long-pedicelled. Rhachis of raceme narrow, 0·3–0 mm. wide; midrib trigonous with a nearly obsolete crest; wings 1–2-nerved, minutely scabrid-toothed, breadth of wings less than the breadth of the midrib, usually about half as broad Internodes of rhachis about as long as the stalked spikelets including the stalk. Pedicels of stalked spikelets 2–3 mm., of subsessile ones 0·3–0·5 mm. Pedicelled spikelets 2·5–3·5 mm. long, about $\frac{1}{3}$ as broad, ovate-lanceolate, short-acuminate. Glume I minute, ovate, obtuse. nerveless Glume II ovate— or elliptic-oblong or ovate-lanceolate acute or obtuse, 3-nerved with thicker intermediate and marginal strips, often suffused with purple, margin adpressed soft-ciliate, dorsally intra-marginally softly adpressed-hairy. Glume III ovate-oblong or broad-lanceolate, acute or subacuminate, 5- (or 7-) nerved, 2·5–3·2 mm. long $\frac{1}{4}$–$\frac{1}{3}$ as broad; margin either scantily or densely and softly adpressed-ciliate, or more rarely beset with soft, ultimately spreading, up to 1 mm. long hairs; dorsal surface either glabrous or softly, adpressedly pubescent between the nerves, or more rarely densely villous along the lateral nerves; hairs often purplish. Glume IV chartaceous, colourless, lanceolate, acuminate, 2·4–2·8 mm. long, slightly shorter than glume III, margins incurved, flaps nearly touching. Palea of gl. IV chartaceous, somewhat shorter, than its glume, lanceolate, acuminate, flaps slightly gaping. Subsessile spikelets similar but somewhat less hairy. Stamens 3; anthers linear, rather more than 1 mm. long.

Two varieties may at present be distinguished:—

var. Thwaitesii sheaths and blades of leaves beset with bulb-based hairs; glume III of pedicelled spikelet densely long-ciliate and dorsally villous along the intramarginal strips, the hairs ultimately spreading (but not mixed with bristle-hairs (Thwaites C. P. 3800).

var. Griffithii, sheaths and blades of leaves glabrous or nearly so; glume III of stalked spikelets softly adpressedly ciliate and dorsally adpressed-pubescent (S. India).

An interesting feature, easily overlooked and owing to its minuteness rather difficult to examine is the palea of glume III (the abaxial flowering glume) of the forms of *P. sanguinale.* The writer has made a detailed study of this palea in the various forms of *Paspalum sanguinale* with a view to discovering whether this feature can be used as a discriminative character, but with negative results. The palea referred to is usually 0·15 to 0·3 mm. high, consists except at its base of one layer of cells, is broad ovate, short-oblong, square, or transversely oblong in outline, with the marginal and submarginal cells free at their upper end, elliptic oblong, pointed, and densely filled with protoplasmic contents. In rare instances the marginal cells elongate into hair-like excrescences. Fig. 21 exhibits various forms of this palea from different localities. The real purpose served by this palea constitutes an interesting problem. (See figs. 23, *a*—*f*.)

PASPALUM AMBIGUUM, DC.

It is a remarkable fact that the Herbarium of the Royal Botanic Gardens at Sibpur possesses no specimen of this species, which is stated in the F.B.I. to occur in Kashmir, Western Tibet and about Simla. An enquiry made at the Dehra Dun Forest School to which the Saharanpur Herbarium has been transferred, elicited the reply that no specimen of *P. ambiguum* can be found there. Collett in the Flora Simlensis remarks that the species occurs at Simla and other parts of the Western Himalaya at attitudes between 5,000 and 10,000 feet.

The specimens marked *P. ambiguum* in the Sibpur Herbarium are all undoubted forms of *P. sanguinale*, chiefly *subsp. cruciatum*. *P. ambiguum* is easily discriminated from *P. sanguinale*, (see fig. 21, *a*, *b*). The *Syntherisma glabrum* of Schrader is a form of *P. sanguinale* and not of *P. ambiguum*, as suggested in the F.B.I. Perhaps some one interested in the botany of the North-west Himalaya will look into this matter and be good enough to present the Royal Botanic Gardens, Sibpur, and the Botanical Department of the University College of Science, Baliganj, Calcutta, with some specimens of this interesting grass

ANASTROPHUS.

(See figs 22 *a*—*d*).

The genus Anastrophus was founded by Schlechtendal in 1850, and based on those species, up to then associated with Paspalum, in which the lowest glume, that corresponding to glume II of Panicum is abaxial instead of being adaxial as it is in Panicum, Digitaria, and the typical Paspalum. Hackel, in Engler and Prantl's Pflanzenfamilien, makes Anastrophus his section III of Paspalum, with most species of which it agrees in the absence of glume I of Panicum. Schlechtendal made *Paspalum barbatum*, *P. compressum*, *P. dissitiflorum*, *P. nemotanthum*, *P. pectinatum*, *P. pellitum*, *P. platycaule*, *P. platyculme*, *P. pulchellum* and *P. setifolium*, most of them Brazilian species, into a separate genus, which he called Anastrophus in consideration of the fact that the position of the spikelets with reference to the rhachis is the reverse of that observed in Paspalum and Digitaria. Regarding the type of nervation of the lower glumes, most of the species of Anastrophus are more closely allied to Paspalum proper, whilst others, such as the Brazilian *A. barbatus* remind one of Digitaria. It appears to the writer advisable to follow Schlechtendal and uphold Anastrophus as a separate genus and not only as a section of Paspalum; this procedure becomes imperative, if we attribute generic rank to Digitaria.

ANASTROPHUS COMPRESSUS (Rasp.) Schlecht.

During the rainy season of 1919 Mr. P. Mukherji, formerly Professor of Chemistry at Presidency College, Calcutta, discovered within the grounds of his residence at Baliganj, a species of Graminaceae, which proved unidentifiable by the F.B.I.

Its characters, as far as they can be ascertained by the aid of the somewhat scanty material at present available, are as follows:—

Lower part of stem creeping. *Culms* erect or the lateral ones ascending, up to 70 cm. long including the inflorescence, slender. *Leaves* crowded near the base of the culm; sheath loosely enveloping the culm, usually coloured ± purple, slightly ciliate towards the mouth; ligule of even width, about 0·75 mm., blade longer than the sheath, linear, up to 30 cm. long, 0·8 cm and less broad, acute, light-green, short ciliate, midnerve not stronger than the side-nerves. *Peduncle* very slender, glabrous, *Inflorescence* of two to five spikes, the two uppermost geminate, the lower ones, when present, at some distance from the terminal pair. *Spikes* very slender, up to 13 cm. long. *Rachis* narrow, flexuous, wings green, about the width of the pale midrib. *Spikelets* sessile, strictly single, alternate, each one reaching slightly beyond the base of the next higher one—alternating with it, flat dorsally, convex ventrally, oblong-lanceolate, acute, 3 mm. or somewhat less long; basal callus not prominent. *Lowest glume* (glume II of Panicum) abaxial, rather flat, lanceolate-oblong, 2·7 × 1·2 mm., membranous, 5-nerved, middle nerve sometimes not reaching the tip of the glume, nerves well defined, green, intermediate spaces hyaline, a narrow dorsal strip of hairs between the median and intermediate nerves, a similar strip between the intermediate and submarginal nerves, margin adpressed ciliate; hairs slender and soft, rather blunt, not knarled, nor glanduliferous. *Next highest glume* similar, but convex, margin inflexed. *Third glume* chartaceous, ultimately ± coriaceous, elliptic-oblong, about 2 mm. long, with a tuft of minute hairs at the apex. *Lodicules* two, obovate, bifid. *Stamens* 3; anthers linear-ob long; thecae linear, purple, separate at base and apex. *Grain* oblong-ellipsoidal, of the length of the palea.

Since the above was written, the writer has observed the same species growing in various other places, and further study has convinced him that it is not a new species, as at first conjectured, but identical with *Anastrophus compressus*, Schlecht, specimens of which from Louisiana are in the Herbarium of the Royal Botanic Gardens, Sibpur. According to the Index Kewensis *Anastrophus compressus* is = *Anastrophus platycaulis* = *Paspalum platycaule* Poir.

Anastrophus compressus, like a number of other immigrant American species, appears to flourish in Bengal and threatens to replace indigenous species of Graminaceae wherever it obtains a foothold. It is spreading rapidly in the Sibpur Botanical Gardens, and the writer has observed it growing on the Baliganj Maidan, in the grounds of the Biological Laboratory and in other localities south of Calcutta A further communication on the species will be published elsewhere.

The writer entirely shares the views of Sir Joseph Hooker and Dr. Stapf as to the wide range of many of the species of the order of Graminaceae.

EXPLANATION OF FIGURES.

PLATE I.

Illustrating the paper on Paspalum, Digitaria and Anastrophus.

Fig. 1.—Spikelet of *Paspalum prolensum*, Trin.

Fig. 2.—Nervation of gl. III of a typical *Paspalum*.

Fig. 3.—Nervation of gl. III of a typical Digitaria.

Fig. 4.—*Paspalum distichum* gl. II: *a*. from Borneo; *bα*. from upper spikelet. *bβ*. from lower spikelet, Sunderbans; *cα*. from upper spikelet, *cβ*. from lower spikelet, Pahang; *d*. from upper and lower spikelet, Karnal.

Fig. 5.—Hairs from gl. II and III of *Paspalum longiflorum*: from, *a*. Aligarh; *b*. and *c*. Dibrugarh; *d*. Madras Presidency; *e*. Narcondam; *f*. Wall. Cat. 8752c.

Fig. 6. Hairs from gl. III of *Paspalum Royleanum*.

Fig. 7.—Crown of hairs on pedicel of *P. Royleanum*: from, *a*. Poona; *b*. Andul; *c*. Rajmahal Hills; *d*. Ceylon.

Fig. 8.—*Paspalum Royleanum*, var. *yunnanensis*: *a*-*c*. from Yunnan: *a* hair from gl. III; *b*. gl. II; *c*. gl. III; *d*. and *e*. from the Nilgiris: *d*. gl. II; *e*. hair from gl. III.

Fig. 9. -*a*-*f* clusters of spikelets of *Paspalum pedicellare*: *a*. lowest branchlet of inflorescence, *b*, *c*, *d*, *e*, second, third, fourth and fifth lower branchlets; *f*. third uppermost of the twenty-one branchlets.

Fig. 10.—*a*. clusters of spikelets of *Paspalum jubatum*; *b*. hairs from gl. II. and III of *Paspalum jubatum*.

Fig. 11.—*Paspalum ternatum*, from Abyssinia: *a*. pair of pedicels; *b*. basal cluster of three spikelets; *c*. gl. II; *d*. gl. III; *e*. hair from gl. III.

BRÜHL: PASPALUM, DIGITARIA & ANASTROPHUS.

PLATE I.

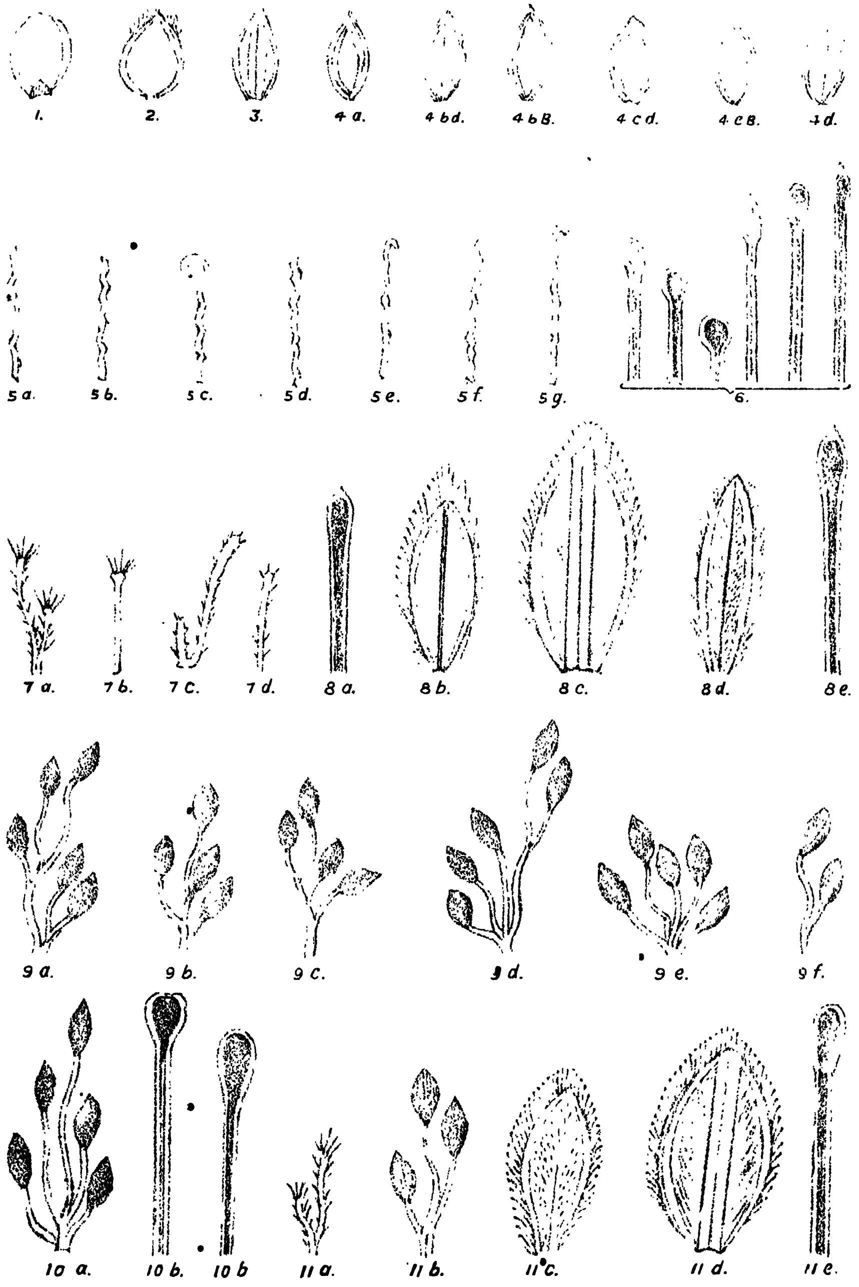


PLATE II.

Fig. 12.—*Paspalum cruciatum*, forma *typica: a–d.* gl. II; from. *a.* and *b.* Dalhousie; *c.* Kurrum Valley; *d.* Khasia, cultivated; *e.* gl. IV, ventral view, from Maflong; *f.* ditto from Jaunsar.

Fig 13.—*Paspalum cruciatum*, var. *setulosa*, from Shillong: *a.* gl. II; *b.* gl III; *c.* gl. IV.

Fig. 14.—*Paspalum commutatum:* gl. II: *a* Baidyanath; *b.* Cudapa; *c.* Afghanistan.

Fig. 15.—*Paspalum extensum:* gl. II: *a.* Mt. Abu; *b.* Gauhati; *c.* Garo Hills; *d.* Lower Bengal; *e.* Southern Burma.

Fig. 16—*Paspalum commutatum*, var *pseudodebilis:* gl. II: from. *a.* Java; *b.* Burma.

Fig. 17.—*Paspalum extensum*, var. *Rottleri:* gl. II: from, *a.* Penang; *b.* Sibpur; *c.* Bengal.

Fig. 18.—*Paspalum ciliare:* gl. III, dorsal view, from. *a.* Baliganj; *b.* Ranchi.

Fig. 19.—*a–c Paspalum pabulare:* gl. II: from three localities, N.-W. Frontier.

Fig. 20.—*Paspalum corymbosum:* gl. III.

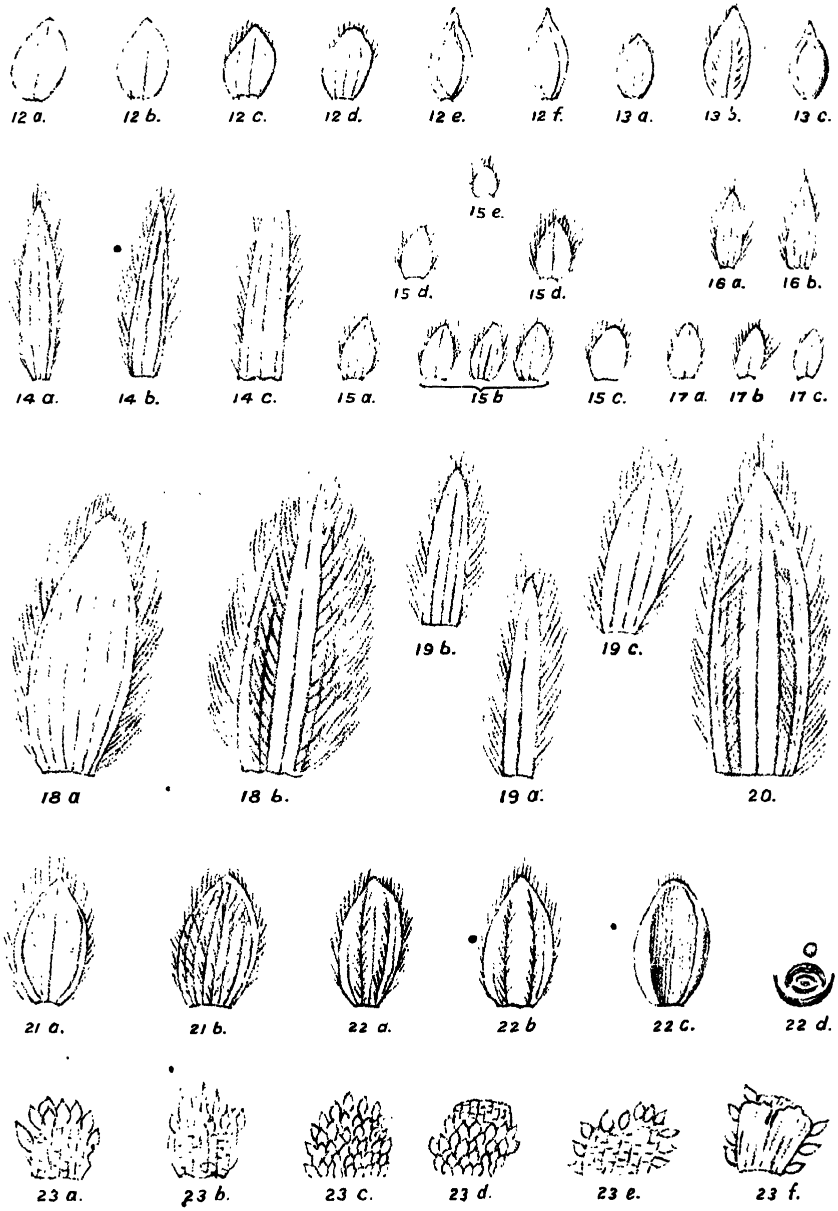
Fig. 21.—*Paspalum ambiguum*, from W. Europe: *a.* gl. II; *b.* gl. III.

Fig. 22.—*Anastrophus compressus*, from Baliganj; *a.* gl. II; *b.* gl. III; *c.* gl. IV; *d.* floral diagram.

Fig. 23.—Palea of gl III of different forms of *Paspalum Sanguinale*, from, *a.* Palni Hills (*extensum*); *b.* Mt. Abu; *c.* Khybar Pass (*pabalare*); *d.* Mussoorie (*commutatum*); *e.* Almora (*ciliare*); *f.* Lachung (*cruciatum*).

BRÜHL: PASPALUM, DIGITARIA & ANASTROPHUS.

PLATE II.



BEHAVIOUR OF PHENYLDITHIOCARBAZINIC ACID TOWARDS VARIOUS TRIHALOGENATED COMPOUNDS.

PRAPHULLA CHANDRA GUHA, M.Sc.,
Sir Taraknath Palit Research Scholar.

Potassium phenyldithiocarbazinate was first described by E. Fischer in 1878.[*] Subsequently, it was employed in the synthesis of some diazole derivatives by Max Busch, by the action of carbon bisulphide, carbonyl chloride, aldehydes, ketones and acid chlorides, etc.[†] Later on, by condensing it with thiocarbimide, the same author jointly with E. Wolpert prepared trizole thiols and also thiodiazole thiols simple by changing the condition of the experiment.[‡] In a series of papers by Max Busch and his co-workers[§] numerous interesting examples of geometrical isomerism have been brought to light in the dialkyl and aryl-alkyl ethers of phenyldithiocarbazinic acid which has been assumed to react by tautomeric change in the dithiol form $PhNHN : C(SH)_2$. The formation of these isomers depends mainly upon the order in which the two alkyl groups are introduced. But, at the same time, cases have been cited in which both the isomers are formed together in one and the same operation, irrespective of the order of introduction of the different alkyl or aryl groups and there are a few instances which preclude the possibility of any strict general rule being framed as to the direction in which the reaction should proceed. To explain the formation of the diazole derivatives and the dialkyl ethers as mentioned above, phenyldithiocarbazinic acid and its potassium salt have been assumed to react in the dithiol form : C(SH) .

The present investigation was undertaken with a view to study the combination of two or more of the above mentioned reactions in one operation, (viz. the action of halogens on the one hand and groups like CHO, COOH, COOEt, NO_2, etc., on the other. As was expected, in almost all the reactions, a mixture of several compounds is simultaneously formed and in one instance two distinct compounds of identical composition have actually been obtained. In a few cases, different compounds formed in the same reaction have been isolated, but in such poor yield that they could neither be analysed nor their properties properly studied.

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[*] *Ann*, *190*, 114.

[†] *J. pr. Chem.*, *60*, 25 [1898].

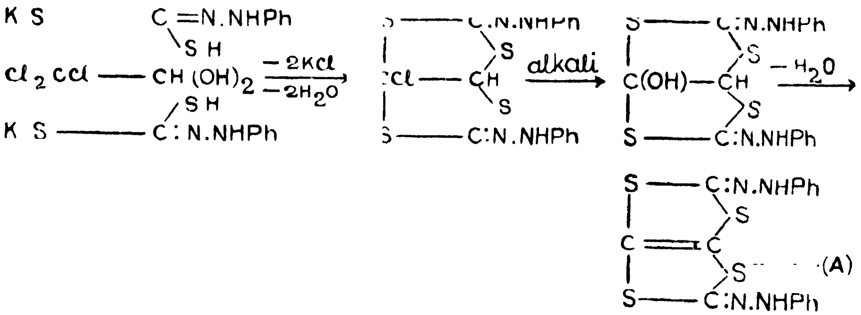
[‡] *Ber.*, *34*, 304 [1901].

[§] Max Busch and Lingenbrink *Ber.*, *32*, 2620; (*J. pr. Chem.*, *61*, 336), Max Busch (*Ber.* *34*, 1119), Max Busch and Hermann Krapf (*J. pr. Chem.*, *84*, 293), Max Busch (*J. pr. Chem.*, *93*, 25).

The most interesting and notable reaction has been found to take place with chloral hydrate which presents the aldehydo group as also the three chlorine atoms to take active part in the course of the reaction.

The course of the reaction may be explained in the following two ways :—

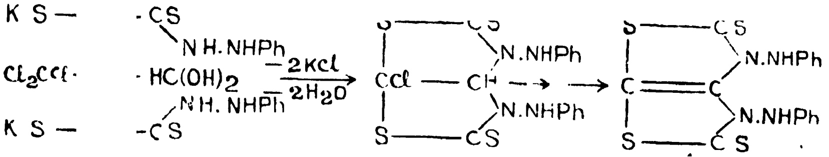
Firstly, by assuming that phenyldithiocarbazinic acid acts by tautomerisation in the dithiol form.



(A)

Here both of the sulphur atoms of the dithiocarboxyl group form members of the ring.

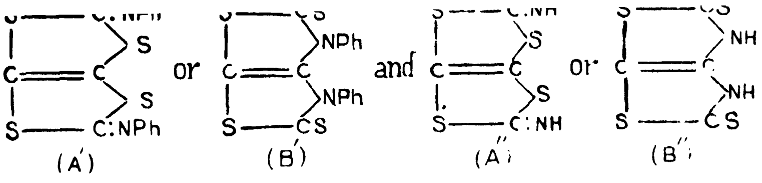
Secondly, without the assumption of the tautomeric formula, thus :—



(B)

Here one atom of sulphur and one atom of nitrogen of phenyldithiocarbazinic acid are two members of one half of the condensed ring-skeleton.

Exactly similar condensed heterocyclic ring compounds have been obtained in the case of Ammonium phenyldithiocarbamate and ammonium dithiocarbamate.



As will be noticed the final products obtained from phenyldithiocarbazinic acid and phenyldithiocarbamic acid do not possess any real or potential mercaptanic group, to render them soluble in alkali, no matter which of the above two formulae A or B is taken into consideration. But, in the case of the simple dithiocarbamic acid, the final product if explained in the light of the formula B'', possesses a potential mercaptanic group; and in reality, it is soluble in alkali and can be precipitated by acid. So, in the opinion of the author it is more expedient to adopt the formulae B, B', B''.

For analogous behaviour of chloral hydrate to form condensed ring, compare its action upon phenylhydrazine.*

Four molecules of potassium phenyldithiocarbazinate react with one molecule of carbon tetrachloride and the compound $(PhNHNHCS)_4C$ is formed with the separation of sulphur, thus: 4 PhNHNHCSSK + CCl_4 = (PhNHNHCSS)4C + 4KCl $\rightarrow$ $(PhNHNHCS)_4C + 4S$.

With bromoform the reaction takes place in an exactly similar way and the resulting compound which is formed is $(PhNHNHCS)_3CH$. With nitrochloroform, however, there is no separation of sulphur, only, three atoms of chlorine combine with three atoms of potassium and the resulting compound is $(PhNHNHCSS)_3CNO_2$.

EXPERIMENTAL.

Chloral hydrate and potassium phenyldithiocarbazinate.

One mol. of potassium phenyldithiocarbazinate was dissolved in water and to it was added an aqueous solution of one molecule of chloral hydrate. The solution was then carefully heated to boiling for about five minutes. A tarry mass, blackish brown in colour, was thus formed which solidified on cooling. The solid mass was well powdered in a mortar and washed several times with water to make it perfectly free from any adhering KCl which was formed during the reaction. An attempt was made to crystallise it from hot spirit, but the hot solution on cooling gave only an oily deposit. On repeating this process for five or six times a reddish-brown powder was obtained which was further purified by crystallisation from a mixture of chloroform and alcohol. Thus obtained, it was brownish-black in colour, softened at 130°C and melted between 157–160°C. It was soluble in benzene, chloroform, methyl alcohol; slightly soluble in ethyl alcohol and perfectly insoluble in water. It was insoluble in alkali.

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* Causse, *Comptes rendus*, *121*, 1029 [1897].

0·0930 gave 0·1673 CO_2 and 0·0284 H_2O. C = 49·08; H = 3·39

0·0787 gave 9·4 c.c. N_2 at 30 and 759·7 mm. N = 13·98

0·1534 gave 0·3656 $BaSO_4$ S = 32·73

$C_{18}H_{12}N_4S_4$, requires, C = 49·48; H = 3·09; N = 14.43; S = 33·00

Two more compounds were isolated from the original solid lump. It was dissolved in chloroform, filtered and concentrated On adding alcohol a small quantity of maroon coloured scaly crystals was obtained which melted at 243 C. On allowing the filtrate to evaporate slowly there separated a few diamond shaped transparent crystals of m.p. 122°C along with the reddish brown powder described above. The latter was obtained in such a small quantity that it could not be analysed; the former one of m.p. 243°C was identical in composition to the compound of m.p. 157 160 C.

0·0498 gave 6·4c.c. N_2 at 35° and 757·5 mm. N = 13·97

Ammonium phenyldithiocarbamate and chloral hydrate.

As in the previous case, equimolecular quantities of the reaction susbtances were boiled in aqueous solution, when a semisolid brownish mass was obtained and the smell of phenyl thiocarbimide was perceptible. The mass was subjected to steam distillation to make it free from phenyl mustard oil. On cooling the whole mass solidified. It was then crystallised several times from boiling alcohol and finally precipitated by alcohol from a chloroform solution. It was blackish-brown in colour, softened at 150°C and melted between 184° to 186°C. The yield of the pure product was very small.

0·0936 gave 0·8124 CO_2 and 0·0286 H_2O. C=53·14; H=3·40
0·0596 gave 3·9 c.c. N_2 at 33°C and 757·8 m.m. N=7·15
$C_{16}H_{10}N_2S_4$ requires, C=53·63; H=2·80; N=7·82.

Ammonium dithiocarbanate and chloral hydrate.

An aqueous solution of the reacting substances was slowly heated and well stirred. At about 65–70°, when the solution grew turbid, it was removed from the source of heat and allowed to cool slowly. After about an hour, a beautiful yellow solid powder settled at the bottom. It was then freed from ammonium chloride by triturating with water in a mortar. It could not be purified by crystallisation, as it was insoluble in almost all the ordinary solvents. It melted at 120–122° with frothing.

It was soluble in alkali and it gave a dull yellow flocculent precipitate with mercuric chloride.

0·0897 gave 0·0747 CO_2 and 0·0125 H_2O C=22·71; H=1·55
0·0766 gave 0·3421 $BaSO_4$ S=61·32
$C_4H_2N_2S_4$ requires, C=23·30; H=0·07; S=62·13

Mono-, Di-, and Tri Chloracetic acids and PhNHNHCSSK.

In aqueous solution, these acids simply act like mineral acids and regenerate the free PhNHNHCSSH from its potassium salt.*

Trichloracetic ester and PhNHNHCSSK.

An alcoholic solution of three molecular proportion of the potassium salt and one molecular proportion of the ester was heated under reflux on the water bath for about half an hour. With the progress of the reaction, the solution became yellow, orange and finally brown. The brown solution was cooled, and on filtering a yellowish residue was left on the filter paper, which was found to be a mixture of sulphur and potassium chloride. After several failures, the following method was found to be the best for the isolation of the pure compounds. The brown solution was heated on the water bath in an open basin for about six to seven hours, when a dark residue was left behind. The dry substance was dissolved in a minimum quantity of acetone and precipitated from it by toluene. This method of precipitation was repeated several times till the oily product became solid. The green mother liquor on keeping exposed to the air for a couple of days, deposited a further quantity of the above substance. It was then boiled with animal charcoal in alcoholic solution and filtered. On concentration and cooling a white crystalline compound of m.p. 196° was obtained. The toluene solution was shaken with a solution of sodium hydroxide, and the orange coloured lower layer of liquid was carefully separated and filtered. On adding acid, a bluish-black substance was precipitated. It was redissolved in alkali and reprecipated by acid in fine powder. This was finally crystallised from dilute acetone in blue needles. Both of these compounds were obtained in very small quantities and so they could not be analysed.

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* Cf. *Trans.*, *115*, 1312 (1919)

Trichloracetic acid and PhNHNHCSSK.

An alcoholic solution of one molecular proportion of trichloracetic acid and three molecular proportion of PhNHNHCSSK were heated under reflux for three hours. The cold solution was filtered and kept aside for slow evaporation. Next day, needle-shaped black crystals of m.m. 171° were found to have separated out. The yield of the compound was too small for analysis.

Carbon tetrachloride and potassiumphenyldithiocarbazinate.

The potassium salt was heated in a sealed tube at 100°, with an excess of carbon tetrachloride, in an alcoholic solution, for about four hours. Next day, a dull yellow crystalline product was found to have separated out from the solution. The tube was opened and its contents filtered. The dry residue was freed from KCl, when it melted at about 164°. The original filtrate gave a second crop of the compound which melted between 145–150°. Both the crops were mixed together and crystallised repeatedly from hot alcohol to get rid of the sulphur. The pure compound thus obtained melted at 170°.

0·0758 gave 0·1585 CO_2 and 0·0385 H_2O C=57·02 ; H=5·66
0·0890 gave 14·4 c.c. N_2 at 32° and 760·1 m.m. N=17·86
0·0585 gave 0·0878 $BaSO_4$ S=20·61
$C_{20}H_{18}N_6S_4$ requires, C=56·49 ; H=4·54 ; N=18·19 ; S=20·87

Bromoform and PhNHNHCSSK.

Four gms. of K-salt and one c.c. of bromoform were heated in a sealed tube with 20 c.c. of alcohol at 100° for about four hours. On cooling, a crystalline dull yellow mass was found to have separated. Next day, the tube was opened and its contents filtered. The residue was a mixture of KCl and free sulphur. The filtrate was evaporated to dryness when a brown pasty mass was obtained. It was then boiled with animal charcoal and filtered, the filtrate again evaporated. On agitating with ether, an oily impurity went into solution and a dirty solid residue was obtained. This was finally crystallised in white rectangular plates from alcohol and melted at 167°. The yield of the pure compound was very small.

0·1096 gave 17·6 c.c. N_2 at 32° and 761·3 m.m. N=17·76.
$C_{22}H_{22}N_6S_3$ requires, N=18·02.

The above reaction was also tried with chloroform and iodoform, but the products obtained in those cases could not be purified.

Nitro-chloroform and PhNHNHCSSK.

An alcoholic solution of the potassium salt was heated on the water bath under reflux with an excess of nitro-chloroform for about 20 minutes. A yellow product was found to have separated out from the reaction mixture. It was cooled and filtered. The yellow residue was freed from KCl and purified by crystallisation from a mixture of chloroform and alchohol. It melted at 131° and was very soluble in chloroform and sparingly soluble in alcohol. It dissolved in hot sodium hydroxide solution.

0·0801 gave 0·1275 CO_2 and 0·0250 H_2O C=43·41; H=3·47.
0·1177 gave 16·5 c.c. N_2 at 28° and 759·9 m.m. N=15·73.
$C_{22}H_{21}O_2N_7S_8$ requires, C=43·49; H=3·46; N=16·14.

I avail myself of this opportunity of expressing my best thanks and gratitude to Sir P. C. Ray for the kind interest he has taken in the work.

CONCERNING THE GRANITES AND PEGMATITES OF THE INDIAN PENINSULA.*

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Geological Survey of India.

The interpretation of the structure and of the mode of formation of the immense outcrops of granite or granitic gneiss observed in this country, constitutes the most important problem of Indian Peninsular Geology. A vast amount of detailed work will be necessary before we possess sufficient data to deal satisfactorily with this question. The object of the present short note is merely to place on record certain conclusions that have suggested themselves to me, either from the available literature on the subject or from direct observation.

For several years, I have gradually been led to regard the majority of granites, not exactly as sedimentary rocks in the usual sense of the term, yet, so to speak, as stratified surface rocks. This is how I expressed myself in the first edition of my *Summary of the Geology of India,* (1907): "The Bundelkhand Gneiss, when the nature and composition of the rock are considered, closely resembles an intrusive granite, but differs from undoubtedly genuine granitic intrusions owing to the enormous area which it occupies. When the Archaean rocks first consolidated, the primordial atmosphere contained in the state of vapour the totality of the water that now forms the ocean, the volatile chlorides, as well as a large proportion of the carbonic acid and oxygen that have now been absorbed by various solid rocks. It is quite conceivable that under the enormous pressure of this primordial atmosphere, molten masses may have spread out over large areas, and on solidifying assumed the granitic form which at later periods could only have been developed under similar conditions of pressure and temperature in the depths of the earth's crust." (p. 6).

At the present day the majority of Indian geologists regard the peninsular granites as invariably intrusive, and invariably newer than the various rocks of the Older Transition systems with which they are in contact, although this view is directly opposed to the conclusions arrived at by the two distinguished scientists, Foote and Middlemiss, who have most specially studied this question in the field.

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* Published with the authorisation of the Director, Geological Survey of India.

In most regions where granitic outcrops have been studied in detail, principally in Europe and in the Rocky Mountains, the granite occupies relatively limited areas, and, in many instances, has evidently reached its present position only at a relatively late geological period. I very much doubt that this can be the case with the extensive granitic spreads, as yet imperfectly known, of Brazil or Canada. In India, judging from all the information at present available, it seems impossible to avoid the conclusion that the granitic gneisses truly occupy their original position over the immense areas which they cover in Bundelkhand and in the Deccan. From the results of direct observation, the earlier, unprejudiced surveyors, principally Mallet, Foote and Middlemiss, regarded the relationship of the Older Transition strata as generally one of normal superposition to the granite. At the present day, most geologists, in India, endeavour to prove that the observed contacts, when not faulted, are invariably intrusive, although not a single example has as yet been adduced of an undoubted apophysis. A great deal more study will be necessary before we can reach a final conclusion. Yet, we are bound to admit that over vast areas, comparable in extent, to the average size of a European State, the Older Transition systems exhibit in their broad outlines, a regularly defined succession, and that they invariably rest on the granitic gneiss. We cannot, at present, avoid the conclusion that the entire Indian peninsula rests upon a continuous layer of granite, and that, when the granitic exposures can be followed uninterruptedly, in every direction, for distances of 150 to 200 miles as in the Hyderabad plateau, what we see is truly the primitive rock in its original position, without any wholesale secondary fusion and without any essentially intrusive character.

South of the immense almost exclusively granitic spread of Hyderabad, the granitic gneiss in the Bellary and Mysore regions still occupies enormous areas, but is traversed by numerous elongate synclinal patches of ancient stratified rocks, partly volcanic, partly sedimentary, whose outcrops succeed one another according to a more or less linear disposition. With the exception of Foote and Middlemiss, all the geologists on the staff of the Geological Survey of India or of Mysore, who have expressed any decided opinion on this subject, have asserted that the granite is newer than the synclinal patches which it carries. We might as well try to prove that a ship is older than the ocean on which it floats.

The oldest strata that rest upon the granitic gneiss contain many rocks, such as sandstones and slates which do not seem to differ from normal aqueous sediments. It is worth enquiring therefore into the possibility of these sediments having been deposited on the floor of an ocean similar to that of the present day, or whether they may be partly anterior to the period of condensation of the aqueous ocean, and formed perhaps in seas or lakes of molten chloride of sodium. The available detailed studies are at present insufficient for definitely answering this question. Certain observations, nevertheless, may afford useful data. The pre-cambrian of peninsular India includes two groups, which are usually sharply distinguished from one another, the "Older Transition," including the great majority of metalliferous mineral deposits, whether as strata or as veins, and the "Newer Transition." The rocks of the older group are almost invariably strongly folded, sometimes thoroughly metamorphosed, though there are local exceptions. Those of the newer group are generally very feebly disturbed, and are mostly quite unaltered; though here again, there are occasional instances in which they are exceptionally altered along certain lines of orogenic disturbance.

Though there is no necessity to exclude the possibility that the sediments of the older group were formed under water, yet the ocean in which they were deposited was evidently at a relatively high temperature. This conclusion seems necessary to account for the vast development of the stratified beds of banded haematite or magnetite accompanied by jasper intercalated amidst those ancient formations. The recurrence, in the "Newer Transition," of these very peculiar banded haematite-jaspers, much less abundantly, it is true, though with identical characters, is of great interest. Indications of the existence of organic life in the "Older Transition" rocks have never been discovered, but undoubted organic remains are known from the "Newer Transition." The presence of *Cryptozoon* in the "Newer Transition" rocks has been ascertained at three spots in the Peninsula, in the Kudapah region, in Chhatisgarh, and in Gwalior. In the latter locality, the limestones containing *Cryptozoon* clearly belong to the same sequence as the banded haematite jaspers from which they are separated by a moderate thickness of strata. Whatever may have been the nature of the ocean in which were deposited the banded ferruginous rocks of the "Older Transition," those of the "Newer Transition" were certainly formed on the floor of an aqueous ocean, the temperature of which was sufficiently reduced to allow the development of organic beings, although this temperature may have been much in excess of that of the sea at the present day. If, under such conditions, the haematite-jaspers could have been formed under water in "Newer Transition" times, there is good reason to believe that the water of the ocean had already condensed when the "Older Transitions" were deposited.

According to stratigraphical observations made in America and Australia, *Cryptozoon* ranges from the pre-cambrian to the ordovician. The pre cambrian age of the "Newer Transition" of India may be regarded as securely established, for the rocks of that group are unconformably succeeded by those of the Vindhyan system which can scarcely be newer than cambrian, if it is not indeed itself partly or entirely pre-cambrian.

We have noticed that, in the Indian Peninsula, the "Older Transition" rocks are almost always intensely folded. They are usually traversed by a network of pegmatites, ranging in thickness from a fraction of a centimetre to a hundred or two hundred metres. These pegmatite veins supply the commercial mica of this country. They exhibit two features of the greatest importance from a theoretical point of view. Firstly, they generally do not show the slightest indication of disturbance subsequent to their solidification They are older than the "Newer Transition" rocks which rest on their eroded edges, but they are evidently later than the final principal phase of corrugation of the penin-

Secondly, there is a total absence of any effects of contact metamorphism upon the enclosing rocks, even when this is a mere friable slate which the most insignificant basaltic intrusion would have baked. These gigantic pegmatites have crystallised therefore at a temperature far too low to produce any thermal alteration in the enclosing rock, a temperature probably inferior to that of a red heat. Nevertheless, when we consider the frequently gigantic size of their crystalline elements, indicating great molecular freedom of the mineral constituents, we are compelled to admit that the solutions from which these crystals were deposited must have been extremely fluid, incomparably more fluid than that from which the granites have crystallised.

Two widely popular notions are negatived by the characters of these pegmatite veins; firstly, the pegmatites are not granitic apophyses; secondly, the rock, which contains no hydrated minerals, does not represent an aqueous solution.*

The pegmatites are not only newer than the "Older Transition" strata, but are even posterior to the period of their final disturbance. They are *a fortiori* incomparably newer than the granite on which the "Older Transition" rocks rest, unless we admit the totally inexplicable notion that the granite is newer than the rocks which it supports. In any case, the notion that substances chemically so intensely active as fluorine and boron, such as abound in many pegmatites could have become concentrated in the end-products of solidification of an alkaline granitic "magma" is as absurd as the popular theories that derive tin, copper, antimony, lead, or gold, by "leaching out" of a previously solidified granite or basalt in which they exist in infinitesimal quantities or not at all. In India, the typical pegmatite veins invariably contain fluorine-bearing minerals: muscovite, apatite, sometimes even flour-spar. When we remember that, even under the ordinary atmospheric pressure, at ordinary temperatures, a very small quantity of fluorine is sufficient to lower considerably the temperatures of solubility of most substances, we may reasonably conclude that the condition of extreme original fluidity indicated by the characters of the pegmatites, is largely due to the presence of that element.

The available data seem to indicate that the pegmatites have originated in an upward direction. So far as I can gather, the solvents have originated at great depths (why or how, I cannot tell, I am merely stating what appears to be a fact), below the granite layer, and already contained, in solution, the heavy metals, before traversing the granite. While traversing the fissures through the granite which already, for a long time previous, had been solidified, they borrowed the elements of those substances (quartz, felspar, mica) which cause them to resemble granites. On nearing the surface, the lowering of temperature promoted the crystallisation, first of the relatively less soluble metallic compounds, such as cassiterite, columbite and tantalite, pitchblende, in a ground-mass of mica, felspar, and quartz; probably at a slightly inferior depth may have been the main zone of crystallisation of ilmenite, monazite, beryl, etc. Nearer the surface, mica and felspar become less abundant, tourmaline often extremely abundant; then we have veins consisting principally of quartz This appears to be the level at which gold and iron pyrites were mainly deposited. At a higher level, the quartz becomes accompanied by calcite; ores of lead, antimony and copper are deposited, as well as barytes. Gradually gypsum appears; we find mixtures of calcite and gypsum, and finally gypsum alone.

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* The hydrogen of unaltered muscovite does not appear to be present in the form of water, but in a state analogous to that of the combined alkali metals. The many practical uses, in the arts or industries, in which muscovite is exposed, without alteration, to high temperatures, sufficiently show that it does not belong to the class of the genuine hydrated substances, the combined water of which, under such circumstances, would be expelled. Even if that hydrogen was derived from water, there is nothing to show that its original home was in the granite.

I do not claim that this succession represents a general rule, but it merely indicates the main plan according to which many veined mineral deposits seem to have originated. If we could follow a gypsum vein downwards through a depth of a few thousand feet, I feel persuaded that, in many cases, we would eventually be led to a vein of pegmatite.

In conclusion, I wish to say that, in spite of the conjectural character of this note, I have considered that it would be worth publishing if only for the sake of suggesting fresh lines of useful enquiry.

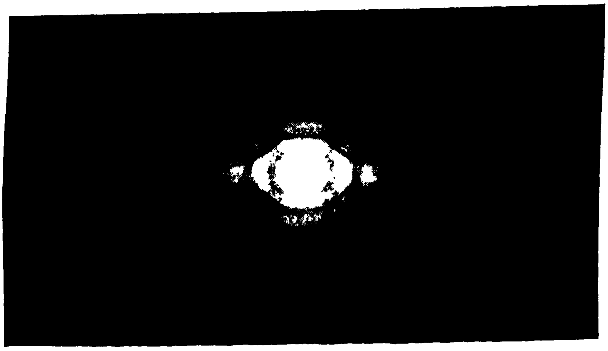

FIG. 1.

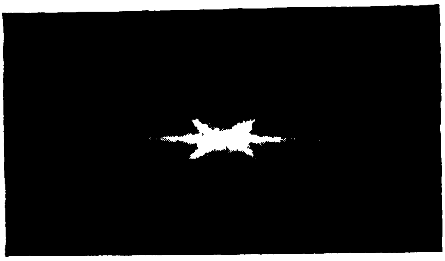

FIG. 2.

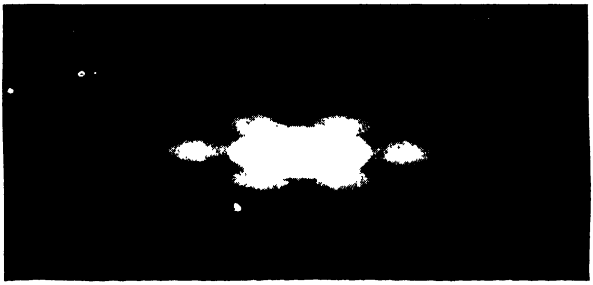

FIG. 3.

Diffraction of Light by Apertures having the form of a Segment of a Circle.

Fig. 1 with aperture greater than a semi-circle, and Figs. 2 and 3 with apertures smaller than a semi-circle).

ON THE DIFFRACTION OF LIGHT BY APERTURES HAVING THE FORM OF A SEGMENT OF A CIRCLE.

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Introduction.

In a recent paper * the author has developed a simple geometrical method of treating diffraction problems and has applied it successfully to find out the form and intensity at any point of the diffraction pattern produced by a semi-circular aperture. In the present communication the same method will be applied to find out the general configuration of the diffraction figures due to a segment of a circle (greater or less than a semicircle). This method has the advantage of enabling the form of the pattern to be deduced readily for a region of any desired area round the focus (which would be a very laborious task if attempted by the ordinary analytical theory) and, even the theoretical determination of intensity can be carried out without any appreciable loss of accuracy and with far less labour; moreover it helps to make a mental picture of how precisely the peculiar configuration of the pattern is related to the particular shape of the diffracting aperture. Figs. 4 and 5 show the geometrical pattern (for segments less and greater than a semicircle respectively) as deduced from the geometrical theory and figs. 1, 2 and 3 are reproductions of photographs of the actual diffraction patterns. Figs. 2 and 3 are due to a segment less than and fig. 1 due to a segment greater than a semicircle. The remarkable similarity between the drawings and the photographs is at once evident.

The Geometrical Theory

The principle of the Geometrical Theory which is discussed in full detail in the paper already referred to can be briefly summarised as follows: instead of proceeding as in the ordinary treatment of diffraction phenomena by expressing the effect at any point in the field in terms of a surface integral taken over the area of the aperture, a considerable simplification is effected by transforming the surface integral into a line integral taken round the boundary, i.e. by regarding the effect at any point in the field as due to a linear distribution of sources of light situated along the boundary. This linear source in its turn is replaced by a finite number of point sources of light generally two, sometimes three or more having appropriate phases situated at certain points on the boundary The position and intensity of these point sources is generally not fixed but varies with the direction of the diffracted light. In other words, corresponding to each point in the focal plane at which the diffraction pattern is formed, there are certain points on the boundary which principally contribute to the luminous effect at the point of observation, and the whole of the diffraction pattern is simply regarded as an *interference* pattern due to a finite number of light sources of *variable* position situated on the boundary. So far from being merely a convenient mathematical fiction, the existence of sources of lights situated at specific points on a curvilinear diffracting boundary may be directly verified by observation or photography. For this purpose the diffracting aperture may be viewed by the aid of the diffracted light only admitted into an observing telescope through a small hole in a screen otherwise completely cutting off the light reaching the focal plane (as is done in the well-known method of Foucalt test). The position of the luminous point in general conform to the following rule: the particular part of the boundary appearing luminous has its normal parallel to the radius vector drawn from the centre of the focal plane to the orifice in the screen through which the diffracting aperture is viewed. When apertures, as of the forms as under discussion in the present paper, are viewed in the foregoing manner the following phenomena are noticed:

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* S. K. Mitra, *Proceedings of the Indian Association for the Cultivation of Science*, Vol. vi, pt. i,

(*a*) For the case of less than a semicircle in general three points on the boundary are seen to be luminous: two of small intensity are situated at the two corners and a third and more intense one on the curved part of the boundary. The position of the latter always conforms to the above mentioned rule namely, the line joining the orifice (through which the aperture is viewed) in the focal plane to centre of the pattern is parallel to the normal drawn to the luminous portion of the boundary.

If however the orifice is placed such that θ lies between ϕ and $\pi-\phi$ or $\pi+\phi$ and $2\pi-\phi$, i.e. between *OD* and *OM* or *ON* and *OE*, (fig. 1, where *A* is the aperture drawn in the focal plane, θ is the angular coordinate of the radius vector *OB* with reference to the line *OC* drawn perpendicular to *DE* and 2ϕ is the angle of the arc) only the two corners appear luminous, because as will be evident from the figure, that for these particular directions no corresponding part on the curved boundary can be found which conforms to the above rule. In the direction $\theta=0$ (or π), i.e. when the orifice lies in a direction perpendicular to the chord *DE*, the whole of the latter becomes luminous together with a luminous point on the corresponding part *C* of the boundary.

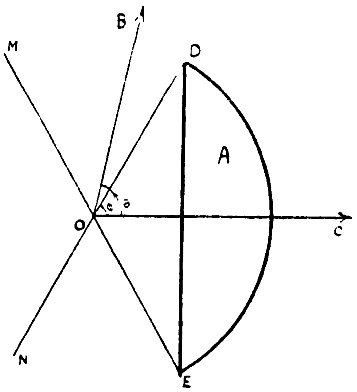

FIG. 1.

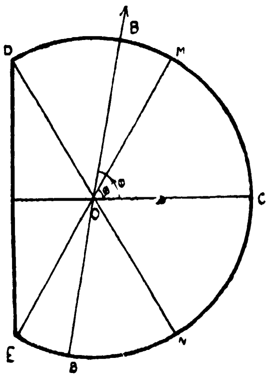

FIG. 2.

(*b*) For the case of a segment greater than a semicircle we might as before distinguish two regions. When the orifice lies within $-\phi$ and ϕ ($\theta > 2\pi - \phi$ and $< \phi$), or within $\pi - \phi$ and $\pi + \phi$ ($\theta > \pi - \phi$ and $< \pi + \phi$) three luminous points, two at the two corners and one at the corresponding point on the arc of the circle is seen, except of course at $\theta = O$ (or π) when instead of the two corners the whole of the chord *DE* is seen to shine up (fig. 2). For other parts of the field, viz. θ lying between ϕ and $\pi - \phi$ or between $\pi + \phi$ and $2\pi - \phi$, four luminous points, two at the two corners and two on the circular boundary are seen, because a little consideration of fig. 2 will show that in these directions two points diametrically opposite to one another (e g. *B* and *B′* in the fig.) can be found conforming to the foregoing rule.

General Configuration of the Pattern.

We can now proceed exactly as has been done for the case of a semi-circular aperture in the paper already referred to, and prove that the whole effect of the aperture might be regarded as a number of sources of appropriate phase and intensity situated at certain points on the boundary. Thus, taking the case of the segment less than a semicircle, let us divide it into narrow strips parallel to the line joining the point of observation to the centre of the pattern in the focal plane as for instance parallel to *OB* (fig. 3). The effect of each strip being reducable to two sources of light at the two extremities that of the whole aperture can be reduced to a linear distribution of sources situated on the boundary of the aperture. The distribution of sources on

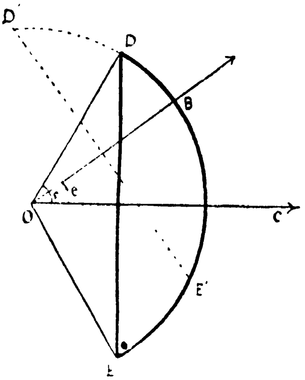

FIG. 3.

the straight portion DE might be replaced by two sources at its two extremities while that on the curved portion might be reduced to a source situated at the point where the phase variation of the contributions to the point under observation is least, i.e. at B. We might regard the amplitude of this latter to be constant for all possible values of θ being, always due to a length of the arc equal to DBE, the effect of the residual and additional portions being replaced by two sources at the two corners. That is, the effect due to DBE (in the direction θ) is equivalent to the effect of the arc $(D'B'E')$+arc (EE')--arc (DD'). The effect of $D'BE'$ is replaced by a source at B, those of EE' and DD' by two sources at E and D respectively. The phases of the contributions from these sources are easily seen to be (with reference to

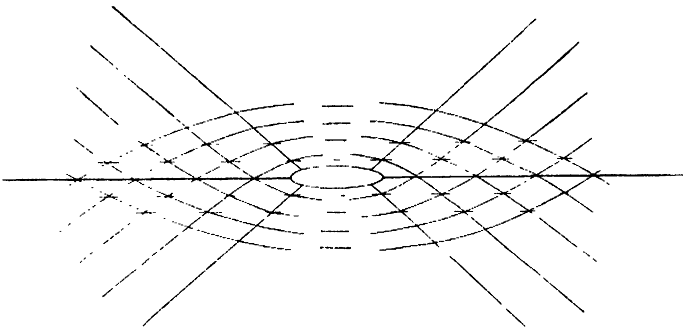

FIG. 4.

the centre of the pattern in the focal plane) $-\delta\cos(\phi-\theta)$ for D and $-\delta\cos(\phi+\theta)$ for E, and $-\delta+\frac{3}{4}\pi$ for B. [$\delta=\frac{2\pi}{\lambda}\cdot\frac{r}{f}\cdot R$, r=length of the radius vector drawn from the centre of the pattern to the point of observation, f=focal length of the lens, and R=radius of the circular arc.]

For B, $\frac{3}{4}\pi$ is the correction due to the curvature of the boundary which can be easily deduced from the case of a circular boundary.

The diffraction pattern can now be considered as due to the mutual interference of these three sources taken two and two together. Thus, in the portion of the field where all the three sources are operating, we have for maximum of illumination due to sources D and B

$$\delta-\delta\cos(\phi-\theta)=\tfrac{3}{4}\pi,\ \tfrac{3}{4}\pi+2\pi,\ \tfrac{3}{4}\pi+4\pi,\ \text{etc.},$$

which, within the specified region gives us a set of branches of parabolas. Similarly E and B gives another set represented by

$$\delta - \delta \cos (\phi + \theta) = \text{const.}$$

Sources D and E together give us a number of straight lines represented by

$$\delta [\cos (\phi + \theta) - \cos (\phi - \theta)] = \text{const.}$$

For the portions of the field where two sources alone are operating the set of straight lines due D and E only are present. When $\theta = 0$, the sources D and E are replacable by a single source at the centre of the chord DE which together with a source at the midpoint of the arc gives us the bright fluctuating horizontal ray seen in the photograph (plate, fig. 2). Fig. 4 is a diagram of the "interference" pattern drawn from the above considerations and it can be seen how closely the drawing reproduces the features appearing in the photograph. Figs. 2 and 3 in the plate are due to segments less than a semicircle. Fig. 3, shows the central portion magnified. For the case of a segment greater than a semi-circle, we can proceed exactly in the same manner and obtain exactly similar results except for the region between OM and OD (or between OE and ON see fig. 2), we get, over and above the sets of branches of parabolas and straight lines, a set of circular arcs due to the interference of B and B'. A drawing of the pattern is shown in fig. 5 and the actual photograph in the plate (fig. 1).

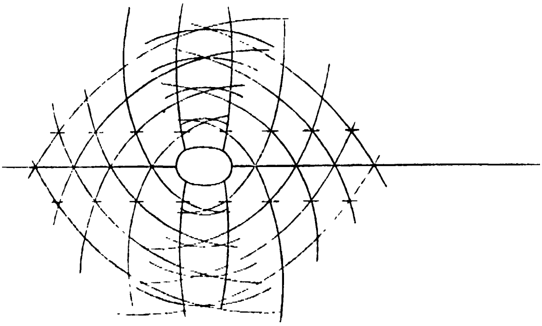

FIG. 5.

Summary and Conclusion.

Diffraction Figures (of the Fraunhofer class) due to apertures having the form of a segment of a circle have been observed and photographed. The general configuration of the pattern has been deduced and their drawings made on the basis of a simple geometrical theory of diffraction which has been developed by the author in a previous communication. These drawings show remarkable similarity to the actual photographs of the diffraction pattern.

The experiments and observations recorded in this paper were carried out in the Palit Laboratory of Physics

NOTES ON THE PANCHET REPTILE.

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Introduction.

The first description of the Panchet reptiles including a labyrinthodont fauna was by the late Prof. Huxley who, in 1865, described a dicynodont reptile from the Panchet beds and named it *Dicynodon orientalis.** Huxley was followed by Lydekker who, in 1879, pointed out its affinities with *Ptychognathus*, Owen,† but as the name *Ptychognathus* had been previously applied to a crustacean genus by Simpson, the name *Ptychosiagum* was proposed by Lydekker in 1889.‡ Cope, however, had in 1870 established the generic name *Lystrosaurus* as distinct from *Ptychognathus*, Owen, the distinctions lying only in the manner of the application of the cutting edges of the mandible to those of the upper jaw §. Subsequent studies have shown that there is no essential difference between *Lystrosaurus*, Cope, and *Ptychognathus*, Owen. Hence according to the rule of priority Cope's name should be used in preference to that of Lydekker, and this method has been followed by Broom and other writers of South African palæontology. Lydekkar, however, stuck fast to the name proposed by him and in his Catalogue of the fossil Reptilia and Amphibia in the British (Museum pt. iv) no mention has been made of the generic name, *Lystrosaurus*.

In the year 1916, I had an opportunity of visiting Deoli, near Asansol, in charge of a party of students from the Presidency College, Calcutta. It was from this locality that most of the specimens described by Huxley and Lydekker were obtained. A few reptilian fossils were collected by the party and this paper contains a description of some of them and of a tibia collected by Mr. B. N. Saha many years ago and now preserved in the Presidency College collection. To this description a few notes have been added and this addition has been rendered possible chiefly by two recent publications on *Lystrosaurus*. In one of them Van Hoepen * described a complete skull of *Lystrosaurus latirostris*, Owen, and in the other a complete skeleton of the same species was described by Watson. †

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* *Pal. Ind., Ser.* iv, Vol. I, pt. 1 (1865), pp. 8–21.

† *Pal. Ind. Ser.* iv, Vol. I, pt. 3 (1879), p. 5.

‡ *Manual of Palæontology*, Vol. II, (1889), p. 1063.

§ *Proc. Amer. Phil. Soc.*, Vol. XI (1870), p. 419.

Description of Fossils.

Humerus.—Huxley established the dicynodont nature of the Panchet reptile mainly from the nature of the humerus. The Presidency College collection includes two humeri belonging to the genus. One of them shows only the distal end with the bridge and the ent-epicondylar foramen well preserved. The specimen is more perfect than any of the humeri of the species hitherto figured. Lydekker suggested the existence of two species of the genus chiefly on the evidence of the humeri. But the humerus on which he relied for his second species was evidently a broken one and the Presidency College collection includes the proximal end of a large left humerus, the form of which presents no difference whatever with the similar parts of the small humeri and identified by him as *Ptychosiagum orientale*. The 'nearly straight border' of the delto-pectoral crest as observed by Lydekker ‡ is owing to the bone having been fractured along the ecto-tuberosity and I think that there is no reason to justify the presence of two species of the genus at Deoli. The proximal part of the large left humerus is figured in the plate (figs. 2*a*, 2*b*).

Tibia.—The Presidency College collection includes a complete tibia collected by Saha and no complete tibia of the Panchet reptile has as yet been described. The distal end of a tibia of the Panchet reptile was figured by Lydekker § and Broom described the tibia of *Ptychosiagum Murrayi*‖. It is clear that both Broom and Lydekker had the same orientation for the bone, namely, the lower end more expanded than the upper one while the tibia described by Watson ¶ is only a fragmentary one. Two complete skeletons of dicynodont reptiles

* *Med. v.h. Trans. Mus.*, Vol. IV, pt. 1 (1913).

† *Rec. Albany Mus.*, Vol II (1912), pp. 287–295.

‡ *Pal. Ind., Ser. iv*, Vol. I, pt. 3 (1879), p. 11.

§ Op. cit. (1879), p. 14.

‖ *Trans. South Afr. Phil. Soc.*, Vol. XI (1900), p 233

¶ *Geol. Mag.* v, Vol. X (1913), p. 258.

have been described showing, among other features, the position of the leg-bones; one of them is the skeleton of *Lystrosaurus* described by Watson * and the other the skeleton of *Oudenodon gracilis* described by Broom †. An examination of both these skeletons shows that the end supposed to be distal by Lydekker is, in all probability, the proximal end and vice versa. The distal end is provided with a very small protuberance, the median shaft is cylindrical and curved, while the proximal end has a faint division into two as in the specimen described by Watson. The median shallow groove, mentioned by Lydekker, is 27 m.m. in length and is situated at an almost equal distance from the two ends (Plate figs. 1, 1*a*, 1*b*).

Rib.—The College collection contains an incomplete rib of the reptile. The portion of the rib preserved is laterally compressed and slightly curved, and bears a thin ridge on its concave side, the ridge becoming more and more indistinct towards the vertebral extremity. Lydekker ‡ compared the ribs of the Panchet fossil with the dicynodont ribs figured by Owen§, but the position of the ridge is quite different. This rib can very well be compared with the post-cervical ribs described by Watson ||. The rib is hollow¶.

Systematic Position of the Panchet Reptile.

An identity of the Panchet reptilian cranium with that of Gordonia has been suggested by Newton **, but a careful comparison of Newton's figures with the Panchet fossils leads one to doubt this identity. The species of *Gordonia* that have been described all agree with the Panchet cranium in having an oval parietal foramen with an interparietal bone in front, but the foramen is placed within a spindle-shaped area and is wedged in between the parieto-squamosal crests posteriorly. The Panchet reptilian cranium, though incomplete, does not appear to show the existence of such a spindle-shaped area, while it also lacks the two prominent

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* *Rec. Albany Mus.*, Vol. II (1912), p. 292.

† *Proc. Zool. Soc*, Vol. II (1901), p. 177.

‡ *Pal. Ind.*, *Ser.* iv, Vol. I, pt. 3, p. 16.

§ *Cat. Fossil Rept. South Africa* (1876), p. 53. These ribs have been subsequently described as belonging to *Eurycarpur Oweni*, Seeley (*Phil. Trans. R.S.*, Vol. 180B., 1889, p. 259).

|| *Rec. Albany Mus.*, Vol. II (1912), Pl. 16.

¶ I am thankful to Mr. Cotter for having drawn my attention to the hollowness of the rib.

** *Phil. Trans. R.S.*, Vol. 184B (1893), p. 444.

parieto-squamosal crests present in *Gordonia.* It may be further pointed out that while describing the scapula of a *Lystrosaurus* sp. Broom expressed a doubt if the Indian reptile was a Lystrosaurus at all [*]. Attention has already been drawn to the papers of Van Hoepen and of Watson published with beautiful figures. A comparison of the Panchet reptilian bones hitherto described with those two figures shows that the Panchet bones very closely agree with the bones of *Lystrosaurus* and that the only other reptile with which some of the bones may be compared is *Oudenodon Bainii* [†] and *O. gracilis* [‡], but the presence of tusks shows that the Panchet reptile cannot be referred to that genus. This comparison has also led me to think that possibly the bone described as the radius of the Panchet reptile is its fibula and that described as its fibula is its real radius. A comparison with the fibula of *Ptychosiagum* (*Lystrosaurus*) *Murrayi* [§] described by Broom also confirms this view. Thus it is clear that though without the discovery of an entire skull, the zoological position of the Panchet reptile cannot be definitely settled, so far as can be judged from the available materials, the Indian reptile can be referred to *Lystrosaurus.*

Habit of Lystrosaurus.

Lydekker pointed out that *Ptychosiagum* was probably a land reptile [||]. Broom thinks that the animal was aquatic and his opinion is based chiefly on (i) the reduction of the interclavicles, (ii) the increase in size of the sternum, and (iii) the shifting backwards of the coracoid and pre-coracoid [¶]. Watson is also in favour of the aquatic habit of Lystrosaurus [**]. Attention may be drawn in this connection to the description of a pelvic girdle of a dicynodont reptile obtained from the Karoo beds of Cape Colony by Broili [††]. Who has observed that "das Ilium dürfte am Becken unserer Form der am meisten charakteristische Knochen sein und zwar durch seine ganz enorm flügelartige rostro-kaudale Verlängerung." The ilium of the Panchet reptile has been described by Lydekker * and the following observation has been recorded by Broili :—

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[*] *Ann. South. Afr. Mus.*, Vol. IV (1908), p. 139.

[†] *Proc. Zool. Soc.*, Vol. II (1901), p. 171.

[‡] Ibid., Vol. II (1901), p. 162.

[§] *Trans. South Afr. Phil. Soc.*, Vol. XI (1901), pp. 223 et seq.

[||] *Pal. Ind. Ser. iv*, Vol. 1, pt. 3 (1879), p. 4.

[¶] *Ann. South Afric. Mus.*, Vol. IV (1908), p. 141.

[**] *Rec. Albany Mus.*, Vol. II (1912), p. 291.

[††] *N. J. f. Min. Geol. u. Pal.*, Bd. I, (1908), pp. 3 et. seq.

"Ganz genau die nämlichen Verhältnisse treffen wir auch bei dem gleichfalls zu den Dicynodontieren gehörigon *Ptychosiagum orientale*, Huxley, aus den unteren Gondwana-Schichten von Panchet bei Raniganj, Bengalen, von dem Lydekker uns ebenso eine Rekonstruktion gibt." The skeleton described by Watson also shows the ilium † which is also anterio-posteriorly elongated. The anterio-posteriorly elongated ilium recalls prominently that of the terrestrial dinosaurian reptiles The ischium of the *Lystrosaurus* is, however, quite different from that of the dinosaur, as the latter is long and thin. The following remark of Watson is also interesting in this connection :

"The extremely long and powerful sacrum is also a surprise in an aquatic animal but it may have served as the insertion of strong longissimi dorsi and sacrolumbar muscles'' .‡

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* *Rec. Geol. Surv. Ind.*, Vol. XXIII, pp. 17 et seq.

† *Rec. Albany Mus*, Vol. II (1912), p. 291.

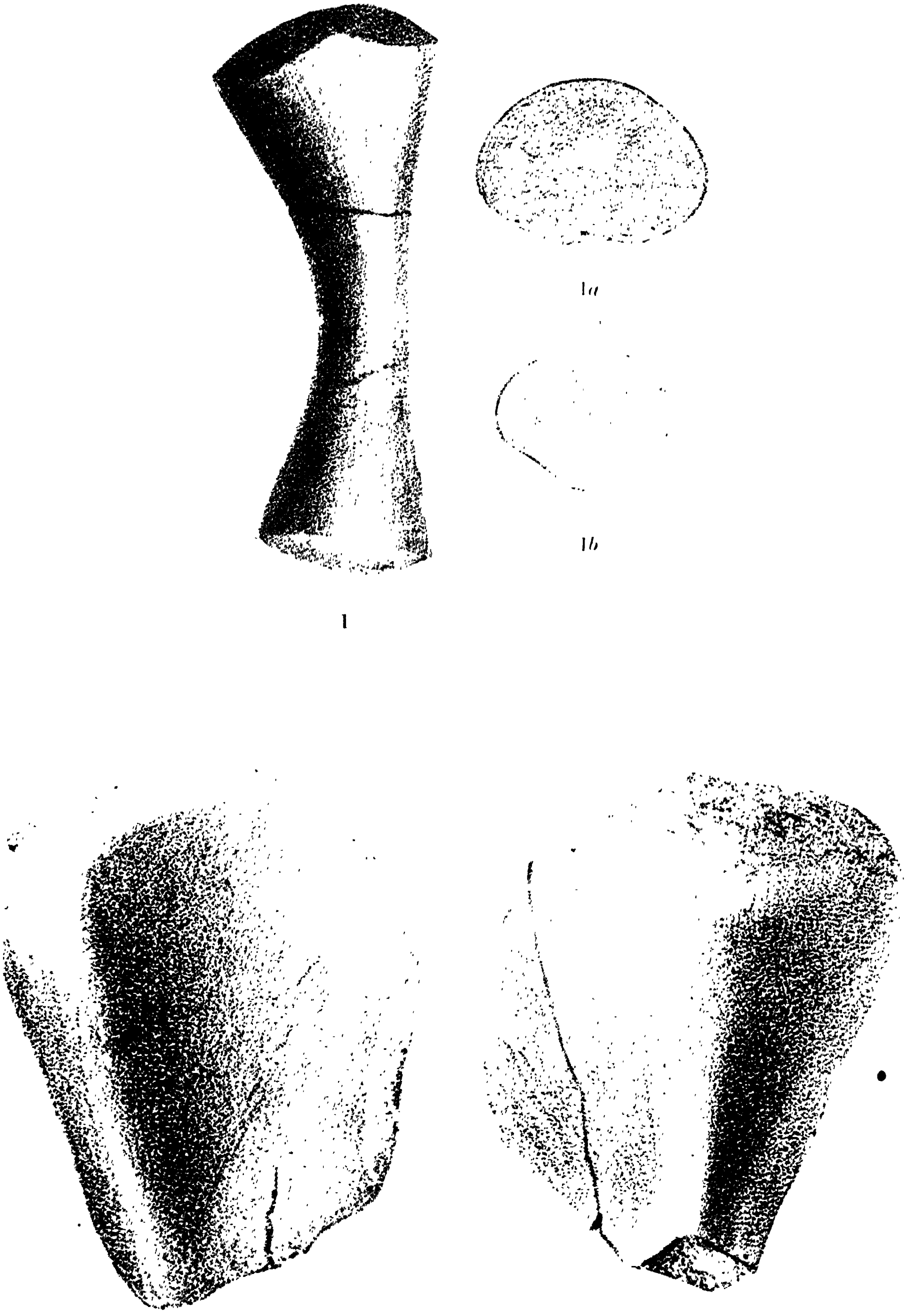
‡ Ibid., p. 292.

EXPLANATION OF PLATE.

Lystrosaurus Orientalis, Hux.

Figs. *1*, *1a*, 1*b*.—Tibia.
,, 2*a*. 2*b*.—Proximal half of left humerus.

(Natural size.)



1 1a 1b 2a 2b

The Panchet Reptile.

Drawn by K. D. C.

ESTIMATION OF MORPHINE, CODEINE AND NARCOTINE IN INDIAN OPIUM.

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Formerly Indian opium had no market in the Western countries. The entire production of the country was consumed by India or by countries to its east. The chemistry of opium and its alkaloids as worked out by the Western Chemists was based mainly on opium grown outside India. There are reasons for getting somewhat different results in experimenting with Indian opium; since the growth, and collection of the different varieties of opium take place under different conditions. The variety of seeds, composition of soil, conditions of weather and methods of cultivation and collection have considerable influence on the chemical constituents of the opium.

The bulk of the Indian opium is produced in the United Provinces. Here the poppy is generally sown from the middle of October to the middle of November. A few days before the drying of the plant only the properly matured capsules containing seeds are given three vertical incisions by the cultivator. Next morning the pinkish white milky sap exuded from the incised places is collected. The process of incision is repeated on the third day to those of the previously incised capsules which are likely to give out any further quantity of the substance. All collections are stored together. This opium passing through different channels comes to the market.

Previous analytical reports regarding the morphine content of Indian opium are not at all favourable. In Allen's *Commercial Organic Analysis* (4th edition, Vol. VI, p. 408), we find "East Indian opium is, as a rule, remarkably weak in morphine, the proportion being sometimes as low as 2·5 p.c, " more commonly between 3·5 and 5 and occasionally as high as 8 or 9 p.c." On the next page occurs the following table :-

Description of opium.	*Morphine.*	
	Crude.	*Pure.*
1. Patna	11·2	8·6
2. Indian (1852-1853)	11·2	4·3

Description of opium.	*Morphine.*	
	Crude.	*Pure.*
3. Abkari	14·2	3·5
4. Behar	10·6	4·6
5. Malwa	14·4	6·1
6. Sind	..	3·8
7. Hyderabad	..	3·2

The above results are much below the morphine strength as required by the Pharmacopoeias; the British Pharmacopoeia requires it to be 9·5 to 10·5 p.c. Probably the publication of these results destroyed all the possibilities of Indian opium finding any market in the western countries. Thus loosing its commercial value it did not form an important subject of investigation for the European chemists and consequently most of their researches were conducted with opium not imported from India.

Within the last few generations there seems to have occurred no fundamental changes here in the method of cultivation or collection of opium so that its morphine content might increase in any way. Several Bulletins from the Imperial College of Science have been issued stating the morphine content of Indian opium to be quite high in many cases and none of them are so very bad as some of the above quoted results. The author has been analysing several thousand samples yearly since the last four years, but he has never come across any sample of Indian opium, which contains 3 or 4 per cent. morphine. Hence naturally the accuracy of those statements becomes doubtful. Perhaps the low results are due to the defective methods of estimation that had been adopted, because a large number of modifications have been published, criticising almost all the hitherto followed methods of estimation.

The climatic conditions of India have very distinct effects on the results of these estimations. The methods prescribed by the British Pharmacopoeia cannot be strictly followed and the following facts are noteworthy in this connection :—

(1) When opium powder and lime are mixed in a mortar with 20 c.c. and then with 60 c.c. water, evaporation of water takes place continually, concentrating the alkaloidal solution in lime water. This evaporation depending on the atmospheric temperature, humidity and the current of air over the surface exposed, is never uniform.

(2) Evaporation during filtration similarly vitiates the results as

above, the more so because the time taken for filtration is never uniform.

(3) The filtrates sometimes do not collect up to 51 c.c. and to get the required quantity the residue has got to be pressed out. This deviation is unavoidable and the liquid thus pressed out is seldom as clear as the first portion. This difficulty can, however, be obviated by taking double the quantities of opium, lime, etc., and then only accepting 51 c.c. for precipitation of morphine.

(4) When ether is transferred on filter paper during filtration a part of it rapidly evaporates leaving a yellowish resinous deposit of alkaloids on the top portion of the upper filter paper and seldom any on the counterpoised lower one. The 10 c.c. of ether recommended by B.P. for washing the filter paper does not completely wash this deposit nor does it make the deposit uniform on two papers. Besides this, the small quantity of ether left on the filter paper after the aqueous mother liquor has passed through, always leaves a residue on the upper filter paper only. The substance deposited by the evaporation of ether is never removed by the subsequent washing with morphinated water. Such residues, however, are always found to be soluble in ether or acid.

(5) Michael Conroy, the discoverer of the method, found that a certain quantity of morphine remains in solution and does not precipitate or crystallise out. To rectify this error he recommended the addition of 0·104 gm. to the actual weight of morphine obtained from 104 c.c. of lime solution of opium. The solubility of the alkaloid in the mother liquor depends upon the temperature and on the other ingredients present there. The above quantity of morphine is supposed to be kept in solution in 104 c.c. of mother liquor at the temperature of cold countries, but where the temperature sometimes rises to more than 110°F. it should retain quite different quantity. Michael Conroy got a deficit of 0·104 gm. of morphine in estimating 1·00 gm. of pure morphine and from a mother liquor which unlike that from opium did not contain much organic matters in solution.

(6) The time recommended for the separation and formation of crystals is 12 hours, but in practice it is kept much longer, generally overnight. There are very few laboratories with more than 12 hours working time. Hence this statement needs modification.

In some laboratories the method of the United States Pharmacopoeia is adopted; which being a complete gravimetric method the errors due to the evaporation during the preparation of solution and filtration are not introduced. Washings with alcohol and ether are recommended which completely removes the residue left after the evaporation of the ether not filtered at the first stage. This method also can hardly be said to be free from defects. When the morphine is taken out of the filter paper for weighment, some of it is always left adhering to the paper. The substance dissolved by the lime water has been taken to be pure morphine which, however, is not the case as is shown below. No correction is made for the retention of morphine by the mother liquor.

The author has described a polarimetric method for the estimation of morphine in opium* and also has done several experiments† showing its advantages over the B.P. process. This polarimetric method is not only convenient as requiring less time to finish than any other method but seems to be more reliable; and the determination is also possible when the sample available is very small in quantity.

Several experiments have been done to compare the results obtained by the polarimetric process with those by the U.S.P. process. A few typical results are given below :—

Sample.	Morphine Pol.	Strength. U.S.P.
Ghazipur	10·2	9·8
Lucknow	10·1	9·7
Cawnpur	9·8	9·6

To ascertain the amount of morphine left in solution after the precipitation of the base by the U.S.P. process the aqueous mother liquor was thrice extracted with 100 c.c. of ether each time to get rid of any codeine that might be still present in solution, evaporated to small bulk, acidulated with 1 c.c. hydrochloric acid, made up to 100 c.c. decolorised with animal charcoal, and examined in 200 mm. tube.

Sample.	$-[a]$	Total morphine in 100 c.c.
1 gm. of morphine hydrochloride (dry) instead of ext. from 10 gms. of opium	−1·3	0·18 gm.
Karmansa	−1·7	0·23 ,,
Ghazipur	−1·8	0·24 ,,
Gowalior	−2·6	0·36 ,,

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* Rakshit, *Analyst*, **43**, 321, 1918.

† Ibid, **44**, 377, 1919.

0·5 gm. of the sample of morphine obtained by U.S.P. process were titrated with N/10 sulphuric acid using litmus indicator; and the neutral or slightly acid solutions were made up to 50 c.c., filtered and examined in 200 mm. tube. The specific rotation was calculated by taking the gross weight to be all morphine and by taking what neutralised acid to be all morphine.

Sample.	Percentage of morphine as obtained by titration.	–[a]	Special rotation	
			on gross weight.	on titrated value.
Ghazipur	97·34	–6·72	–116·5	–119·5
,,	97·34	–6·80	–118.0	–121·2
,,	96·8	–6·72	–116·5	–120·4
Pure morphine	....	....	–127·5	–127·5

0·5 gm. of morphine obtained by the U.S.P. process was taken in a 250 c.c. Erlenmayer flask shaken with 50 c.c. of freshly made lime water filtered through tared double filter washed with lime water till the filtrate on acidification ceased to give any precipitate with Meyer's reagent. The amounts dissolved and undissolved by lime water were usually determined gravimetrically. The lime water solution was made up to 250 c.c. with lime water and examined in 200 mm. tube and specific rotation was calculated taking the weight of morphine to be what was dissolved by lime water. The following results show that the matters dissolved by lime water are not all morphine:—

Sample.	Percentage of morphine as obtained by the dissolution in lime-water.		Specific rotation of morphine.
Pure Morphine	100·0	–1·4	–60·68
Benares opium	86·7	–1·0	–49·9
Ghazipur ..	93·8	–1·0	–46·2

Estimation of Codeine.

Among the opium alkaloids codeine is next in importance to morphine, so its determination require some consideration. The common method now adopted for this purpose is the one described by Andrews* which is not only a very lengthy process, but also is not free from defects. The resinous mass separating on the addition of sodium salicylate solution, which has been said to contain thebaine, is practically free from any alkaloid and consist chiefly of salicylic acid and colouring matter. The results obtained are never concordant and the codeine isolated is seldom free from other alkaloids; the latter could be ascertained by polarimetric examination. Codeine obtained by the process has been titrated with N/10 sulphuric acid, made up to 50 c.c. examined in 200 mm. tube and specific rotation is calculated accepting the corrected weight obtained from titration.

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* *Analyst*, **36**, 489, 1911.

Sample.	eight of codeine obtained		Specific Rotation.
	Gross.	Corrected by titration.	
Codeine pure ..	0·200	0·200	−137·5
Ghazipur .	0·200	0·167	−124·8
Lucknow ..	0·180	0·142	−130·0
Gowalior .	0·108	0·100	−130·0
Simla ..	0·104	0·088	−132·0
Cawnpur ..	0·089	0·068	−128·0
Azamgarh ..	0·160	0·120	−128·0

The above figures clearly show that the codeine value obtained by titration does not represent the actual amount of codeine present. Consequently an attempt has been made to work out a plan for the estimation of codeine which could give more reliable results than the above. Among the opium alkaloids that are more or less soluble in ammonia, morphine, pseudo-morphine, thebaine, meconidine, codamine, laudanine, laudanidine, protopine, narceine, and porphyroxine require consideration. The rest however may be practically regarded as insoluble. Of these morphine, pseudo-morphine, laudanidine and narceine are practically insoluble in ether, and the remaining thebaine, meconidine, codamine, laudanine, protopine and porphyroxine are much less soluble in ether than codeine; with the exception of the last one these are present only in a very small quantity in Indian opium. Porphyroxine and thebaine are, among those that are more or less soluble in ether and optically active; but the latter is either present in a small quantity or almost entirely absent in Indian opium.

An aqueous extract of opium was precipitated with strong ammonia (sp. gr. 0·880), filtered, shaken up three times successively with ether, the etherial extract shaken up with 1% solution of acetic acid and the acid solution of the alkaloids separated and neutralised with calcium carbonate. Porphyroxine was not eleminated by this process and when the alkaloid was re-extracted with ether after making the solution alkaline with caustic soda the residue left on evaporation of the ether did not crystallise like codeine and on warming with a little dilute hydrochloric acid gave the usual red colour of porphyroxine.

It was then observed that when porphyroxine was heated with dilute hydrochloric acid on steam-bath, it rapidly changed into a red colouring matter and this on again treating with caustic alkalies gave a precipitate which remained undissolved by ether. Under such conditions codeine did not undergo any change. After numerous trial experiments depending on this process of eleminating porphyroxine from codeine the following scheme has been found to give considerable satisfaction.

Twenty grams of opium and 200 c.c.'s of water are taken in an Erlenmeyer's flask, shaken for 3 hours and filtered. 100 c.c. of the filtrate are added to 20 c.c. of strong ammonia contained in a similar conical flask, shaken for one hour and filtered. One hundred c.cs. of this filtrate are taken in a 500 c.c. stoppered separator, thrice extracted with ether using 100 c.c. each time. The etherial extracts were filtered into another 500 c.c. separator, the separator and the filter paper was rinsed with 200 c.c. of ether. The etherial extract is twice shaken for 10 minutes with 10% solution of hydrochloric acid using 25 c.c. at a time. The two acid extracts are taken in a basin and evaporated to dryness on steam-bath. The residue thus obtained is generally dark-pink coloured ; it is dissolved in 30 c.c. of distilled water by warming a little on steam-bath, transferred into a separator, 50 c.c. of ether and 10 c.c. of 10% solution of caustic soda are then introduced, and shaken for 10 minutes. The aqueous layer is taken out into another separator and the extraction is repeated twice more with similar quantities of ether. The etherial extracts are dried over calcium chloride, filtered, evaporated to dryness and dissolved in 10 c.c. N/10 sulphuric acid. It is then filtered, made up to 50 c.c. and examined in 200 mm. tube.

Percentage of codeine in opium

$$= \frac{\text{Reading in Ventzke scale} \times 100 \times 0{\cdot}3468 \times 1{\cdot}2 \times 10}{2 \times (-137{\cdot}5)}$$

The following are the results by the above process :—

	Sample.	Percentage of Codeine.
1.	Using 2 gms. of codeine instead of 20 gms. of opium	99·6
2.	Ghazipur opium	1·63
3.	,, ,, (duplicate)	1·84
4.	Gwalior ,,	1·52
5.	Lucknow ,,	0·88
6.	,, ,, (duplicate)	0·79

Narcotine.

Before the effects of quinine on malarial fevers were well established narcotine was used in the treatment of such cases. Now it has lost all its importance as medicine and hence very little attention is paid to the quantity present in opium. Several investigators have estimated it in opium, but they have not done so by a common well-established method. In Henry's *Plant Alkaloids* (pp. 205) we find that "Narcotine is usually estimated by extracting the dried opium with dry ether or benzene, and shaking the solution with ammonia, which removes narceine. The narcotine left on distilling off the ether or benzene is dried and weighed." Allen* states that "narcotine may be extracted from dried opium by ether or benzene." A reference to the properties of opium wax † will show how erroneous these directions are. The narcotine thus obtained will not only contain small quantities of other alkaloids, but also the bulk of the opium wax. Narcotine is said to occur in opium in the free state and as such is extractable by solvents like benzene or ether. It was thought worth while to find out if the alkaloid could be extracted by solvents as it is or whether it could be better done by previous treatment of the opium with some alkali. Accordingly (1) 10 gms. of opium powder, (2) a mixture of 10 gms. of opium powder and 5 gms. of freshly slaked lime, and (3) a mixture of 10 gms. of opium powder and 5 gms. of ammonium carbonate were successively extracted with 100 c.c., 50 c.c. and 50 c.c. of benzene respectively. The benzene extracts were shaken up vigorously with 100 c.c. of strong ammonia for half an hour, which formed an unseparable white emulsion. This was transferred into a basin and heated on steam bath till the benzene was completely evaporated leaving yellowish soft residue on the top of aqueous ammonia. The residue was filtered and washed till the filtrate ceased to give any odour of ammonia. The residue with the filter paper was heated with 100 c.c. of 1% hydrochloric acid on steam-bath when a pinkish solution was obtained This acid solution was filtered into a separator and the residue washed with a small quantity of water; 100 c c. of benzene was introduced into the separator, the solution was made alkaline with ammonia, then faintly acid with dilute acetic acid, and shaken up well for 10 minutes. The extraction with benzene was repeated using 50 c.c. The benzene extracts were filtered, dried and weighed.

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* *Commercial Organic Analysis*, 4th ed., Vol. vi, p. 401.

† Rakshit, *Analyst*, 1918, 321.

	Narcotine.
10 gms. pure opium (Benares)	3·6 gm.
,, ,, ,, +5 gms. slaked lime	3·3 ,,
,, ,, ,, +5 ,, $Am_2 CO_3$	4·6 ,,
,, ,, ,, (Gwalior)	2·7 ,,
,, ,, ,, +5 gms. ed lime laks	2·8 ,,
,, ,, ,, +5 ,, $Am_2 CO_3$	3·8 ,,

The aqueous solutions after extractions with benzene were further acidulated with dilute hydrochloric acid and treated with Meyer's reagent and tested for optical activity. Only that from the ammonium carbonate gave a good precipitate with Meyer's reagent but did not rotate the plane of polarisation.

These results clearly proves how much work is still left to be done to reveal the true chemistry of Indian opium.

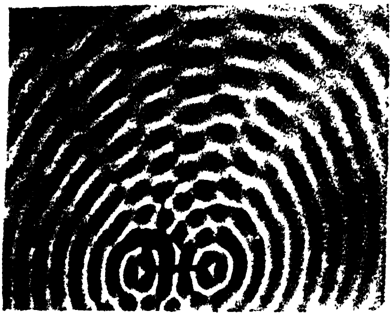

Fig. 1.

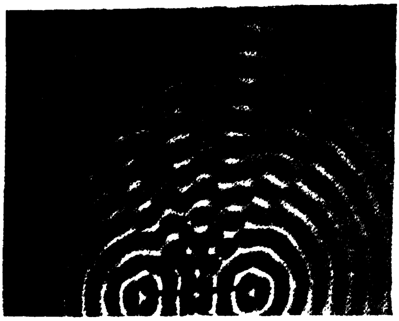

Fig 2

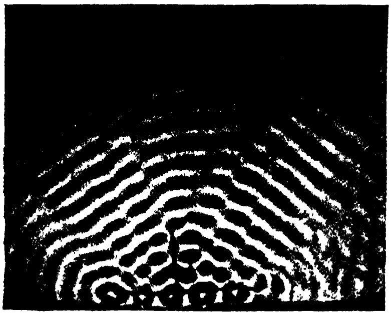

Fig. 3.

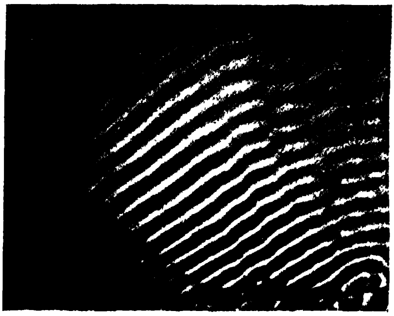

Fig. 4.

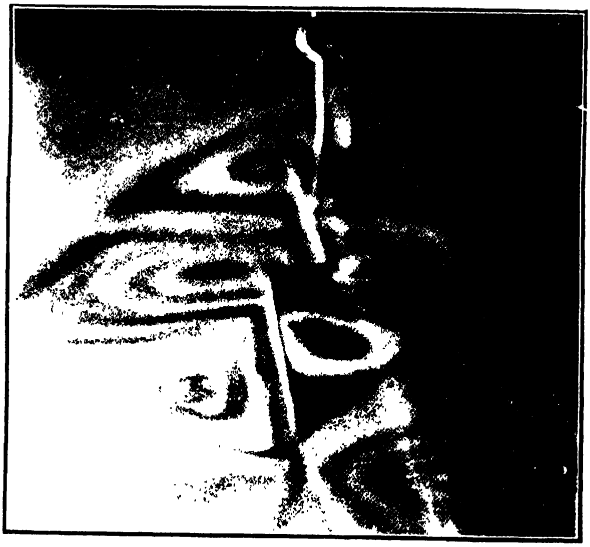

Fig. 5.

Ripple Motion.

SOME EXPERIMENTS ON RIPPLE MOTION.

GOVERDHAN LAL DATTA, M.A.,
Palit Research Scholar, Calcutta University.

1. *Introduction.*

Ripples on mercury were produced and photographed by Vincent as early as 1897 and 1898; in his experiments he tried various forms of sources and obtained very beautiful and instructive results. The subject has continued to attract the attention of other physicists, who have worked on the same or different lines. A fairly comprehensive bibliography of the literature on ripple-motion will be found in the paper by F. R. Watson in the *Physical Review* for February 1916. Various analogies with optical phenomena have already been illustrated by the results obtained by these experimenters, but there are some of special interest, which have not yet been studied, and it was with a view to study them, that the present experiments were undertaken. In this paper, one of the experiments described deals with the analogy of the problem of the effect of groove-form on grating spectra, and of the theory of R. W. Wood's Echelette Grating. There is also one aspect of even the simplest interference experiments with ripples which has not received attention so far. The experimenters have generally confined their attention to what happens at fairly considerable distances from the sources. It has appeared to the present author that a close study of the phenomena observed in the immediate neighbourhood of the sources would be of interest. This has been done for simple cases. The paper also describes the phenomenon of the surface movement ot the water in the ripple tank which has been observed in the course of the experiments.

The apparatus used for the production of ripples is an electric-motor-vibrator similar to that described by Prof. Raman in the *Physical Review* for November 1919. The rotation of the shaft of the motor causes a horizontal beam to vibrate up and down. A dipper of the desired form can be attached to the beam, exciting ripples on the surface of the water in a tank with glass-bottom. Instantaneous photographs are not taken, but on the other hand, light is allowed to fall from an arc on a disc of cardboard pierced with a hole, and mounted on the shaft of the said motor, so that the incident light passes only once through the hole during the complete vibration of the dipper. After passing through the hole in the disc, it falls on a mirror which reflects it on to the other mirrors, so that eventually it comes up through the glass-bottom of the tank. Since the disturbance is synchronous with the light, hence, if a screen be placed above the tank, ripples can be focussed and can be seen on that constantly, wherefrom they can be photographed.

2. *Ripple Analogy with the Echelette Grating.*

Vincent worked with a comb, i.e. a large number of points at equal distances, and found that he could obtain spectra corresponding to the optical case of diffraction by a large number of narrow slits in line. The present author has also studied the ripples produced by a comb of six points and obtained three spectra, one in the middle, and one on either side (Fig. 3 in the Plate.) In all actual optical gratings, however, we are confronted with the effect of groove-form on the distribution of energy in the spectra. Wood's Echelette grating brings out this point clearly, when he gets the whole energy concentrated into one or two spectra at the most on one side of the central image. The question arises, is it possible to get the whole energy concentrated into one side-spectrum in the case of ripples or not? For this purpose, a grating of special form was constructed. A horizontal rod has soldered to it a number of plane strips set obliquely at an angle of 60° nearly, at a distance of 2 cms. from each other. The portion which dips in water is shown in the diagram.

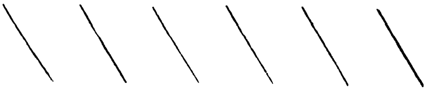

FIG. 1.
Form of Echelette Grating for Ripples.

The strips in all were six in number. When properly adjusted with the least possible sideway motion and not too vigorous a vibratory motion, the whole of the energy is thrown into one spectrum to the left of the centre just as in Wood's Echelette Grating (Fig. 4 in the Plate).

If we look at a point just near the source in the lower edge of Fig. 4, we see the corrugated waves of Lord Rayleigh which however soon join up to form a plane wave-front. To test that the spectrum obtained is a real one, the speed of the motor was changed thereby changing the wave-length. It was found that the angle of the spectrum, became smaller or greater in accordance with the formula,

$$n\lambda = K \sin \theta$$

When λ is made smaller, θ becomes smaller.

Now if the sideway movement is not suppressed and also the vibratory motion is vigorous (the amplitude great), a faint spectrum appears on the other side of the centre.

3. *Ripple-forms in Interference Experiments near the Sources.*

Fig. 2 in the plate shows what happens when two sources move in the same phase. Just around the two points the wave-fronts are practically circles surrounding the sources but are to some extent deformed on account of the interference. As we proceed further and further from the sources, it becomes less and less easy to distinguish the wave-fronts due to the two sources separately. It will be noticed that any maximum across the central line (bisecting the join of the sources and perpendicular to it) falls just against the minimum at the side or in other words midway the successive maxima on the two sides. But away from the central line that is no longer the case. Any one maximum does not come in midway between the maxima on either side of it, but is nearer to one than to the other, so much so that in very oblique directions the successive maxima very nearly join up. In addition, the picture shows some disturbance even along the stationary lines and faint oblique lines connecting up the contiguous maxima.

So far as the author is aware, the exact form of the wave-fronts in the neighbourhood of two interfering sources, has never been discussed, though the question of the form of the lines of energy-flow in this case has recently received some attention. * It seems worthwhile to make an attempt to account mathematically for the features noted above, i.e. the maximum coming against the minimum in the central region and almost joining up with contiguous maxima in oblique directions. The problem has been dealt with mathematically on the following lines. The displacement at a particular point is due to both the sources and the amplitude varies inversley as the distance from the source. If the distance of the point from one source be called r_1, and from the other source r_2, the amplitude will be $\frac{c}{r_1}$ due to one, and $\frac{c}{r_2}$ due to the other. The displacement at a point being a periodic function can be represented by

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* R. W. Wood, *Phil. Mag.*, *18*, 250 ; and Max Mason, *Phil. Mag.*, *20*, 290

$$Z = \frac{c}{r_1} . \sin 2\pi \left(\frac{t}{T} + \frac{r_1}{\lambda} \right) + \frac{c}{r_2} . \sin 2\pi \left(\frac{t}{T} + \frac{r_2}{\lambda} \right)$$

Considering the whole field at a particular epoch, $t=0$, this expression is simplified to

$$Z = \frac{c}{r_1} . \sin 2\pi . \frac{r_1}{\lambda} + \frac{c}{r_2} . \sin 2\pi . \frac{r_2}{\lambda}$$

where Z is the displacement of a water particle perpendicular to the plane of the paper, c is a constant, λ is the wave-length, r_1 and r_2 the distances of the point under consideration from the sources. At a very great distance, c/r_1 and c/r_2 become almost equal, and the case becomes of ordinary interference, but at nearer points, the difference between c/r_1 and c/r_2 is sufficiently large in comparison to the quantities involved, and hence must be taken into account. Within the region shown in the diagram No. 2, the values of the displacements at different points are obtained from this expression by substituting the values of r_1 and r_2 which are measured actually by scale. Any suitable value of λ can be chosen which must remain the same for the whole field. C the constant does not come in, for all we require are the comparative values of the displacements at different points for plotting curves The procedure is as follows. The origin is taken midway between the sources and radial lines are drawn from it quite close to each other, and within the region equidistant points on the lines are marked very near each other. It is found that along a particular line, the displacement is positive for some distance, and then becomes negative, remains so for some distance and again becomes positive and so on, similarly on the adjacent lines Generally on the adjacent lines, the displacement is like at points at the same distance from the centre. But it so happens in some cases that the positive displacements on one line come against the negative displacements on the next. This change seems to be a rapid one, for the distance between two such lines is extremely small. In the present case for which the calculations were made, the angular distance between two such lines is of the order of one degree. The displacement being determined for points lying within an angular range of 40° from the central line on either side, the contour lines of displacement for zero and other values were plotted. The zero lines as seen from the diagram curve upwards almost suddenly at a point near the line where the displacements become opposite on adjacent lines. The strip in the diagram where the zero-lines curve upwards gives the region of minimum disturbance. Along this region at no point does the displacement become great. Although the displacement is actually zero at only a few points, it is very small all along in this region.

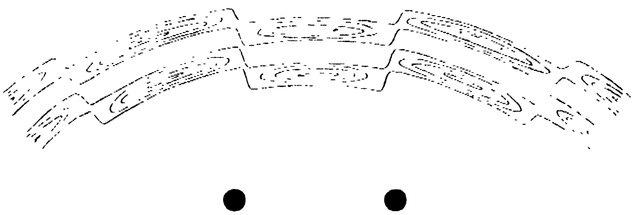

DIAGRAM 2.
Form of the contour lines of equal disturbance in an interference field

Thus it is seen that as the zero lines curve up, they enclose a region of displacement of same sign. Within the zero-lines enclosing the positive displacements, the various contour lines are plotted. They are all closed curves; very narrow and small for great displacement, and sufficiently large for comparatively small displacements. As the diagram shows, near the central region one maximum just comes against the minimum at the side, but as the distance from the central line is increased, the distance between two maxima decreases and the maximum is no longer directly against the minimum (centre to centre). The diagram which has been drawn from the calculated values is very similar to the ripple-forms actually obtained (Fig. 2 in the Plate). Thus it is seen that by considering c/r, the amplitude due to the source, it can be explained why the maxima tend to fall in a line in oblique directions. Moreover it shows that the maximum is compressed in the centre and rather drawn out on the sides. It shows also that the smaller the value of the wave-length, the nearer are the so-called stationary lines to each other.

Three Sources.

Nearer the sources the phenomenon is similar to the one just considered in the case of two sources, but at a distance there seems to be a tendency to form a spectrum which becomes very clear in the case of six sources. We have two sets of interferences—one lying midway between the others (Fig. 1 in the Plate).

Six Sources.

Near the source (Fig. 3 in the Plate) ordinary interference as in other cases happens, but at a moderate distance the spectra become quite marked—three spectra separate out One in the middle, and two on the two sides; the wave-front becoming almost plane.

4. *Circulatory Motion on the Surface of Water.*

As has been remarked already in the beginning, an interesting surface movement of water was noticed. It is found that water is thrown into vortices, big and small, as Figure 5 in the Plate shows. Near the ends of the dipper the movement becomes very vigorous as if the ends are, 'sucking in' the liquid. Whether this circulatory motion of water is due to the side way movement of the source producing ripples or whether it is due to the terms of the second and higher order in the equation for the propagation of waves (as is shown by Lord Rayleigh in the case of circulatory motion produced in the air-pipes) is not yet known. To photograph this motion, the surface of water was sprinkled with lycopodium powder through a muslin piece to get rid of clusters of particles. Light was coming as before through the bottom but this time it was not intermittent. The surface of water was directly focussed and the photograph taken.

In the end, I cannot help expressing my deep gratitude to Prof. Raman at whose hands I have received constant encouragement and guidance in the study of these ripples.

IONISATION OF ELECTROLYTES IN SOLUTION.

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Van t-Hoff discovered that aqueous solutions of electrolytes show an abnormal osmotic pressure. As is well known, for these solutions the equation $PV = iRT$ holds good, where P is the osmotic pressure and V the volume of solution containing a gram-molecule of the electrolyte. Now, there are two possible ways of explaining this abnormality. The one, that holds the field now is the theory of Arrhenius, according to which an electrolytic molecule dissociates into ions, and $i = (1-\alpha) + n\alpha$ where α is the degree of dissociation and n the number of ions into which a molecule dissociates. He discovered an independent method of determining α the degree of dissociation based on the observation that the molecular conductivity of a solution increases with dilution, and attains a maximum at infinite dilution. Hence if the conductivity be due to ions only,

$$\alpha = \frac{\mu_v}{\mu_\infty}.$$

The law of mass-action is evidently applicable to this case of chemical equilibrium and the following dilution law

$$\frac{\alpha^2 c}{(1-\alpha) c} = k,$$

should accordingly hold good. In the case of weak electrolytes, e.g. acetic acid, this equation holds good very well, but it breaks down hopelessly in the case of strong electrolytes like KCl. This is quite evident from tables 1 and 2.

Various attempts have been made to explain this anomaly of strong electrolytes but none based on the idea of progressive dissociation with dilution has been in the least successful.

An entirely different mode of explanation is however possible similar to that advanced for the deviations exhibited by actual gases from the theoretical gas law $PV = RT$. We may assume that only ions exist in solutions of strong electrolytes, and i would have been always equal to n but for the inter-ionic forces in solution.

Let A_v be the work required to separate completely the ions constituting a gram-molecule of sodium chloride when the dilution is V. The ions in the solution are of course endowed with a kinetic energy of translation, the distribution of energy being governed by Maxwell's Law. As the work done in escaping from the electrical field inside the solution, must be derived from the kinetic energy of the ions, only those ions can escape which have a kinetic energy greater than the work to be done. These are the active ions, and they only conduct electricity. The fraction of the total number of ions which are active at dilution V is given by the expression

$$nN.e^{-A_v/nRT}.$$

Therefore,

$$\mu_{v_1} = nN.e^{-A_{v_1}/nRT} \text{ and } \mu_{v_2} = nN.e^{-A_{v_2}/nRT},$$

or $$\frac{\mu_{v_2}}{\mu_{v_1}} = e^{A_{v_1} - A_{v_2}/nRT}, \quad \text{or} \quad A_{v_1} - A_{v_2} = n\,RT \log \frac{\mu_{v_2}}{\mu_{v_1}}.$$

At infinite dilution, the ions are beyond one another's sphere of attraction,

$$A_{v_\infty} = 0, \text{ hence } A_{v_1} = nRT \log_e \frac{\mu_\infty}{\mu_v} = nRT \log_e \frac{1}{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Here α may suitably be termed the activity-coefficient as in Arrhenius's original memoir.

The problem before us is to determine the value of A_v,—the work required to separate the oppositely charged ions constituting a gram-molecule, to an infinite distance apart from their mean distance in the solution when the dilution is V. The existence of inter ionic force demands a regular arrangement of the ions in the solution. The distribution is not random as in the case of perfect gases. The way in which this random distribution will be modified is given by a theorem due to Boltzmann. Let us take a large number say x of instantaneous views of a certain region of the solution containing N ions, and in each view, observe the position and signs of all the ions which are present in it. In a certain number x' of these x views, the N ions will be found in small equal volumes dv_1, dv_2 .. dv_N situated at the points P'_1, P'_2 .. P'_N. Let this be P_1 configuration. In another number x''

of views, these same N ions will be found in positions P''_1, P''_2 . . P''_N. Let this be the P'' configuration. On a purely random distribution $x' = x''$. But if there be an interionic force

$$x' = x'' \, e^{-(A' - A'')/kT}.$$

where A' and A'' are the respective mutual potential energies of the ions in the two configurations P' and P''. We thus find that the configuration which will come into view the maximum number of times must be characterised by the minimum potential energy, the probability of the other configurations containing greater energies falling off very rapidly.

The question arises what is the most stable arrangement of ions in a solution of uni-univalent salts like potassium chloride. In nature, we find a ready illustration of this stable arrangement. The sylvine crystal which has been thoroughly investigated by Prof. Bragg, by his X-ray method of analysis, is the nearest analogue of an aqueous solution of potassium chloride. In the crystal of potassium chloride, molecules do not exist. The positively charged potassium ions and the negatively charged chlorine ions are arranged alternately in a cubic space lattice. We may assume that in a solution of uni-univalent salts, the arrangement of ions is similar to that in the sylvine crystal. The work required to separate the two nearest opposite ions in such a system to an infinite distance is equal to $\frac{E^2}{Dr}$ as a first approximation. We simply imagine that the two nearest opposite ions form a completely saturated electrical doublet. r, the distance between the ions, can be easily obtained, for if there be $2N$ ions, in V c.c., the whole solution is made of $2N$ unit cubes, where r is the linear dimension of unit cube.

$$2Nr^3 = V \text{ or } r = \sqrt[3]{\frac{V}{2N}};$$

E is of course the absolute electric charge

$$4 \cdot 7 \times 10^{-10} \text{ E.S.U.}$$

N is Avogadro's Number $6 \cdot 16 \times 10^{23}$, and D the dielectric constant of the solvent. There are N such doublets; hence

$$A_v = \frac{N \, . \, E^2 \, . \, \sqrt[3]{2N}}{D\sqrt[3]{V}}$$

Therefore

$$\log_e \frac{\mu_{v_2}}{\mu_{v_1}} = \frac{N \cdot E^2 \cdot \sqrt[3]{2N}}{D \cdot 2RT} \left\{ \frac{1}{\sqrt[3]{V_1}} - \frac{1}{\sqrt[3]{V_2}} \right\}$$

This equation contains not a single arbitrary constant, and on substituting the actual values, becomes

$$\log_{10} \frac{\mu v_2}{\mu_{v_1}} = 1 \cdot 616 \left\{ \frac{1}{\sqrt[3]{V_1}} - \frac{1}{\sqrt[3]{V_2}} \right\}$$

In uni-bivalent salts like the chlorides of calcium, barium and strontium, I assume, an arrangement identical with the fluorspar crystal According to Prof. Bragg, the calcium atoms are arranged in a face-centered cubic lattice, while the fluorine atoms occupy the centres of the small cubes. The equation for this type of salts in aqueous solutions becomes

$$\log_{10} \frac{\mu_{v_2}}{\mu_{v_1}} = 2 \cdot 97 \left\{ \frac{1}{\sqrt[3]{V_1}} - \frac{1}{\sqrt[3]{V_2}} \right\}$$

where μ_v is the equivalent conductivity and V the equivalent dilution. The agreement between the observed and calculated values will be evident from table 3.

It is interesting to note that equation (1) is the limiting case of Kohlrausch's empirical relation

$$\mu_v = \mu \quad - k \sqrt[3]{c}.$$

For $\log \frac{\mu_\infty}{\mu} = \frac{\mu v \mu_a}{\mu_\propto}$ where μ_∞ is nearly equal to μ_v,

or $1 - \frac{\mu_v}{\mu_\infty} = k' \sqrt[3]{c}$, or $\mu_v = \mu_\propto - k \sqrt[3]{c} \ldots . (1a)$.

The influence of temperature on ionisation.

In the equation for uni-univalent salts

$$\frac{N \cdot E^2 \cdot \sqrt[3]{2N}}{D \cdot \sqrt[3]{V} \cdot 2RT} = \log_e \frac{\mu_\infty}{\mu_v} = \log_e \frac{1}{a},$$

the left hand expression contains in the denominator D and T, and they almost counteract each other's influence. According to Drude, the variation of the dielectric constant of water with temperature is given by the formula.

$$D_t = D_{18}\{1 - 0 \cdot 00436\ (t - 18) + 0 \cdot 0000117\ (t - 18)^2\}.$$

D_{100} according to this formula is 52·6.

Even the slight diminution in the value of the activity coefficient with rise in temperature can be quantitatively predicted as is evident from table 4.

Influence of Solvent.

The only property of the solvent which enters into the dilution law is its dielectric constant. It follows at once, that the activity-coefficient α of the same electrolyte in different solvents, at the same temperature are equal, when the dilutions satisfy the relation

$$D_1\sqrt[3]{V_1}=D_2\sqrt[3]{V_2}=D_3\sqrt[3]{V_3}\ldots, \text{ etc.}$$

Prof. Walden has investigated the electrical conductivity of tetraethylammonium iodide in some forty solvents, and deduced the above rule as an empirical generalisation from his exhaustive data. The degree of accuracy with which the above equation reproduces the conductivity at various dilutions for non-aqueous solutions will be evident from table 5.

The electrical conductivity of acids and bases in aqueous solutions.

For aqueous solutions of strong acids like hydrochloric or nitric acids or for strong bases like potassium or sodium hydroxides, the ratio $\frac{\mu_v}{\mu_\infty}$ actually observed is much greater than that derived from the equation

$$\frac{1\cdot616}{\sqrt[3]{V}}=\log_{10}\frac{\mu_\infty}{\mu_v}=\log_{10}\frac{1}{\alpha}.$$

This irregularity however disappears as we pass from the aqueous to the non-aqueous solutions of strong acids. The variation of the molecular conductivity of HCl in methyl alcohol for example, as shown in the previous table, can be exactly calculated from equation (1).

It has always appeared remarkable, that the conductivities of hydrogen and hydroxyl ions in water are not of the same order of magnitude as those of other ions. For example $\mu_{H^\cdot}=318$; $\mu_{OH}=175$, μ_∞ for other ions never exceed 70 at 18°. In non-aqueous solutions of acids again this abnormally high couductivity is not absorbed. It thus appears probable that the abnormally high value of the ratio $\frac{\mu_v}{\mu_\infty}$ in this case, has something to do with the abnormal mobility of the hydrogen ion in aqueous solution. In fact $\frac{\mu_v}{\mu_\infty}$ is not, here, a real expression for the activity-coefficient of the ions. Fortunately, we have an independent method of determining α, under the condition that no electric current should pass through the solution. Ellis has determined the electromotive force of cells of the type H_2 gas/ HCl/ Hg_2 Cl_2/Hg, and obtains the values of α given in table 6 from the well-known equation

$$E_2 - E_1 = 2RT \log_e \frac{a_1 c_1}{a_2 c_2}.$$

where the concentrations of the hydrochloric acids in the cells are c_1 and c_2 respectively. He assumes that at exceedingly high dilution, α obtained from E.M.F. method is the same as $\frac{\mu_v}{\mu_\infty}$.

The agreement between the values of α calculated from equation (1), and those obtained from the measurements of Ellis are quite good, and it thus appears probable that the real activity coefficient of hydrochloric acid is the same as that of uni-univalent salts. Arrhenius suggests that the exceptionally great conductivity of hydrogen and hydroxyl ions in water, is due only to the fact, that these are the ions into which water is electrolytically decomposed. The observed conductivity of hydrogen and hydroxyl ions in water is the additive effect of two separate and independent processes namely: (1) the transference of electricity by the convection of charged bodies; and (2) the transference of electric charge through molecules of water by the alternate processes of dissociation and recombination during impact with hydrogen or hydroxyl ions. In the first process, only the free ions take part in the transference of electricity and their number is given by the equation

$$N' = 2N \,.\, e^{-A/nRT}.$$

The conception of the second process is similar to that imagined by Grotthuss. It appears probable that when a hydrogen ion strikes against a molecule of water the latter in some cases undergoes dissociation. It may well be, that the hydrogen atom of the water molecule which is farthest from the point of impact, shoots off as a charged particle, the hydroxyl residue combining with the impinging hydrogen ion to re-form water. If the processes of dissociation and recombination be instantaneous, the electric charge E, appears to be carried instantaneously through a distance which is proportional to the diameter of the water molecule. Hence,

$$\mu_v = a\,(U_H + U_{A'}) + C \ldots \quad (2),$$

where a is the true activity-coefficient at dilution V and U_H, the real mobility of the hydrogen ion. C is the conductivity due to the second process.

Again, $$\mu_\infty = (U_H + U_{A'}) + C \ldots \quad (3),$$

C has the same value in both these equations, for we are always dealing with a gram-molecule of the acid, and the number of impacts are the same, because all the hydrogen ions, both free and vibrating, are effective here Equations (2) and (3) contain only two unknown terms U_H and C, and can therefore be easily solved. From the conductivity data of solutions of hydrochloric acid, U_H comes out equal to 152·4 and C equal to 197·8 at 25°C. The values of U_H and C having once been determined it is possible to calculate the molecular conductivity of any strong acid at any dilution from equation (2). The agreement will be apparent from table 7.

In the case of acids of intermediate *strength-transition electrolytes* and weak acids, we have as in Arrhenius's original theory both undissociated molecules and ions in solution. The law of mass action in its original form is not however applicable to charged bodies. It requires modification in the sense, that only the free ions take part in the equilibrium. They, only, are free to move, and hence have the chance of colliding with one another. Thus if x be the fraction of a gram-molecule of acid which has undergone dissociation, the law of mass action is not expressed by the equation $\frac{x^2}{(1-x)V} = k$, but by the equation

$$\frac{(ax)^2}{(1-x)V} = k, \ldots \quad (4),$$

where a is the activity-coefficient at the ionic concentration $\frac{x}{V}$. In the case of very weak acids like acetic acid, x is generally very small, $\frac{x}{V}$ becomes always negligibly small, and a which is given by the equation (1a):

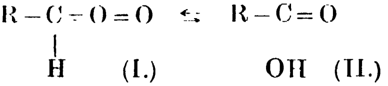
$$a = 1 - k\sqrt[3]{c} = 1 - k\sqrt[3]{\frac{x}{V}}$$

is very nearly equal to one at all dilutions. For weak acids, therefore, Ostwald's Dilution Law holds good. For transition electrolytes, a is not always equal to 1, and here equation (4) must be applied as such.

Here,
$$\mu_v = xC + \alpha x\,(U_H + U_{A'})$$
$$= xC + \left(1 - k\sqrt[3]{\frac{x}{V}}\right)x\,.\,(U_H + U_{A'}),$$

only x is unknown, and its value can be easily found out by the method of successive approximation. Table 8 shows that the values of k obtained from equation 4 are fairly constant, and hence it truly represents the behaviour of transition electrolytes.

This clear cut division of electrolytes into two classes appears unnatural, and it is probable that in the case of weak acids, we are dealing with a dynamic equilibrium between a non-polar compound which does not dissociate at all, and a polar compound which always dissociates completely. The tentative suggestion might be put forward, that even in carboxylic acids there is an equilibrium between the forms



$$\underset{\textstyle \mathrm{H}}{\mathrm{R-C}}\mathrm{-O=O}\ \text{(I.)} \rightleftarrows \underset{\textstyle \mathrm{OH}}{\mathrm{R-C}}\mathrm{=O}\ \text{(II.)}$$

On salt formation, only the polar form II. exists, for a metal does not directly attach itself to a carbon atom. Hence salts of weak acids are also strong electrolytes. The generally rapid oxidation of the aldehydes and the existence of the peroxide of benzaldehyde point to the possibility of the existence of form I. Indeed, it is Hantzsch's theory of pseudo- and acid-forms, pushed to its logical conclusion.

The theory developed above, also accounts for the activity coefficients of salts in mixed solutions in a satisfactory manner. Here we have the empirical rule of Noyes, that " the conductivity and freezing point lowering of a mixture of salts having one ion in common, are those calculated on the assumption, that the activity-coefficient of each salt is the same as that of its solution where the concentration of either ion, is the sum of the equivalent concentrations of the positive or negative ions present in the mixture.'' Let the equivalent dilutions of two uni-univalent salts be V' and V''. On mixing the volume becomes $V'+V''$, and there is a redistribution of ionic arrangement until the entire solution is uniform. The dilution is of course

$$= \frac{V'+V''}{2}\,,\ \text{and}\ \log\frac{1}{\alpha} = k\sqrt[3]{\frac{2}{V'+V''}},$$

where α is necessarily the activity-coefficient of the ions in the mixture.

This is the rule of Noyes. The condition of isohydry follows at once for if $V' = V''$

$$\log \frac{1}{a'} = k \sqrt[3]{\frac{1}{V'}} = \log \frac{1}{a''} = k \sqrt[3]{\frac{1}{V''}} = \log \frac{1}{a} = k \sqrt[3]{\frac{2}{V' + V''}}.$$

The conditions of equilibrium in a salt solution in presence of a solid phase, supply further evidence against the validity of Arrhenius's theory of electrolytic dissociation, according to which there is an equilibrium between the solid salt and the un-ionised salt molecules in the dissolved state; the latter in its turn are in equilibrium with the ions into which they dissociate. The law of mass action is thus applicable to these processes, and the following principles of solubility have been developed by Nernst and Noyes respectively :—

(1) The concentration of the undissociated portion of the salt, present as a solid phase, has the same value in dilute solutions of other salts, as it has when it is present alone.

(2) The product of the concentrations of the ions of a salt present as a solid phase has the same value in the dilute solutions of other salts as it has when it is present alone. This is known as the constancy of the ionic solubility-product.

From his investigations on the solubility of certain silver salts, Arrhenius himself concluded, that the first principle is not even approximately true. It is remarkable however, that the ionic solubility product of these silver salts remains exactly constant over a large range of concentration of the added salts. The explanation is very simple. The first principle—constant concentration of the undissociated salt molecules—cannot be true, because they really do not exist. The second principle is true because only the active ions take part in this heterogeneous equilibrium. Of course, in all these cases where the solubility increases in presence of a salt with a common ion, we generally deal with a formation of complex ion, and the phenomena is a complicated process of chemical equilibrium.

We finally come to the question, with which we started, the significance of the Van 't-Hoff coefficient *i*. If the osmotic pressure be due to the kinetic bombardment of the solute molecules, the well-known Virial Theorem of Clausius becomes very suitable for application to salt solutions. Here the analogy with imperfect gases is complete and we may write

$$PV=\frac{2}{3}\textit{ kinetic energy}-\frac{1}{2}\textit{ potential energy.}$$

The potential energy is, of course, the work required to separate the ions of a gram-molecule to infinite distance,

$$\text{or } PV=n\,RT-\frac{1}{3}\,A=n\,RT-\frac{1}{3}\,n\,RT\log_e\frac{1}{\alpha}$$

$$=RT.n\left\{1-\frac{1}{3}\log_e\frac{1}{\alpha}\right\}$$

According to Arrhenius $i=(1-\alpha)+n\alpha$. Both equations had to the same limiting value of $i=n$.

In tables 9 and 10 the observed values of i are the weighted mean of the freezing point data of a large number of investigators for uni-univalent and uni bivalent salts respectively. They are taken from a paper by Noyes. It is evident that the equation

$$i=n\left\{1-\frac{1}{3}\log_e\frac{1}{\alpha}\right\},$$

based on the Clausius Theorem, agrees best with experimental results. The Arrhenius equation fails conspicuously in the case of ternary electrolytes. Jones attempted to explain this anomaly on his hydrate theory according to which the effective concentration is much greater than the apparent concentration because the water molecules combine with the ions. There is no doubt that ions are hydrated, but there is some inherent improbability in the fact that in dilute solutions, about 150 water molecules must be made to combine with an ion, in order that the difference between the observed values of i and those obtained from the Arrhenius Theory may be explained. It is simply unnecessary, in veiw of the validity of the Clausius Theorem for salt solutions. Indeed for uni-univalent salts, Noyes obtained the empirical law

$$i=2-b\sqrt[3]{c}.$$

$$\text{Now } i=n\left\{1-\frac{1}{3}\log_e\frac{1}{\alpha}\right\}=n.\left\{1-\frac{1}{3}.\frac{k}{\sqrt[3]{V}}\right\}=n-\frac{1}{3}\,n\,k.\sqrt[3]{c}=2-b\sqrt[3]{c}.$$

Thus the empirical law of Noyes follows at once from the Virial Theorem. A complete quantitative theory of dilute electrolytic solutions can thus be developed by taking into consideration the inter-ionic forces in solution.

TABLES.

TABLE 1. Acetic Acid.

V in litres	μ	100α	$100k$
13·51	6·086	1·570	·001845
434·2	33·22	8·568	·001849
3474·0	86·71	22·36	·001855

TABLE 2. KCl.

$V=$	10	50	200	1,000	10,000
A	·8453	·9152	·9532	·9781	·9931
K	·5817	·1975	·0971	·0437	·0143

TABLE 3.

NaCl; eqn., $\log \frac{\mu_{v_2}}{\mu_{v_1}} = 1·616 \left\{ \frac{1}{\sqrt[3]{V_1}} - \frac{1}{\sqrt[3]{V_2}} \right\}$

$V=$	10	20	1,000	5,000
μ calc. from μ_{100}	93·1	96·5	106·4	108·0
μ obs.	92·75	96·1	106·5	107·8

$BaCl_2$; eqn., $\log \frac{\mu v_2}{\mu v_1} = 2·97 \left\{ \frac{1}{\sqrt[3]{V_1}} - \frac{1}{\sqrt[3]{V_2}} \right\}$

$V=$	10	20	1,000	5,000
μ_v calc. from μ_{100}	92·0	97·3	116·2	119·5
μ_v obs.	92·2	96·8	116·9	119·8

TABLE 4. KCl. T = 100°.

$V=$	12·5	100
α calc.	82·6	90·9
α obs.	82·6	91·1

TABLE 5. NaI in Methyl Alcohol.

$V=$	10	50	1,000	10,000
μ_v calc. from μ_{100}	55·8	66·0	77·8	81·9
μ_v obs. (Völlmer)	57·1	65·2	78·8	83·3

HCl in Ethyl Alcohol; Dielectric Constant = 21·7 (Drude).

$V=$	10	20	50	200	1,000
μ_v calc. from μ_{100}	31·9	36·5	41·7	48·0	53·0
μ_v obs. (Partington)	31·9	36·4	41·6	48·2	53·3

TABLE 6.

$V=$	5	10	20	50	100
α from eqn 1.	·805	·844	·875	·906	·924
α from E.M.F. for HCl.	·818	·843	·874	·918	·940
$\frac{\mu_v}{\mu}$ for HCl.	·896	·925	·944	·962	·975

TABLE 7.

	$V=$	20	100	2,000
HNO_3	μ_v obs.	393·3	406·0	417·0
	μ_v calc.	393·1	404·8	415·0

	$V=$	100	400	1,600
Napthalene-β-sulphonic acid	μ_v obs.	367·4	374·9	377·4
	μ_v calc.	367·2	372·4	376·2

TABLE 8. Eqn., $\frac{(ax)^2}{(1-x)V} = k$.

O-Nitrobenzoic Acid.

V	X	A	K
32	·3810	·9204	·00621
128	·5987	·9407	·00617
1024	·8861	·9664	·00627

Dichlor-acetic Acid.

V	X	A	K
16	6337	·8835	·0535
32	7400	·9007	·0534
64	·8318	·9174	·0540
128	·9021	·9319	·0550

TABLE 9. Uni-univalent Salts.

$V=$	2	10	50
i obs.	1·804	1·880	1·925
i Clausius Theorem	1·806	1·887	1·934
i Arrhenius Theory	1·748	1·843	1·906

TABLE 10. Uni-bivalent Salts.

$V=$	10	20	50
i obs.	2·620	2·690	2·771
i Clausius Theorem	2·616	2·695	2·773
i Arrhenius Theory	2·356	2·470	2·594

RELATIVITY OF TIME AND SPACE.

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1. It is necessary to preface this paper with a brief explanation; Most of us have read Sir Oliver Lodge's remark that the most surprising thing in connection with the recent discussions on relativity was that any one should profess knowledge of what the Theory of Relativity really means. In view of this remark, it might seem to be almost a work of super-erogation, for any one to come forward to give an account of that theory, which shall not appeal only to specialists. Nor shall I essay it, in its entirety. My purpose is rather to explain, so far as I can, what it is not, rather than what it is, and if I attempt anything further, it will be to state how I have tried to picture it to myself, rather than what the final pronouncement of Science on the Theory will be.

2. I should like to premise further, that much of what has appeared on the subject, sometimes under sensational headlines is in newspapers altogether misleading. That quite a number of educated men should ask whether Newton's "Theory of the Universe" has been upset is perhaps natural in view of the comparative ignorance that prevails even among educated men as to what that theory or any other theory attempts to do, but it is necessary, at the outset for us to be assured that so far, none of the past achievements of Science has been rendered illusory, although we are beginning to regard the concepts of Space and Time in a some what changed aspect, the change being however, by no-means, revolutionary.

3. These concepts of time and space have been almost instinctively treated by Physical Science, as primordial—baffling all attempts at analysis or definition. Metaphysicians have, no doubt found, in such attempts much scope for ingenuity but all such have proved wholly infructuous.

In a sense, indeed, the views that have been held regarding Space and Time have had a remarkable tendency to persistence. "All our knowledge," says Maxwell, "both of time and space is essentially relative." We cannot describe the time of an event except by reference to some other event, or the place of a body except by a reference to some other body. He points out further that "when a man has acquired the habit of putting words together, without troubling himself to form the thoughts which ought to correspond to them, it is easy for him to frame an antithesis between the relative knowledge and a so-called absolute knowledge and to point out our ignorance of the absolute position of a point, as an instance of the limitation of our faculties. But any one who will try to imagine the state of mind conscious of the absolute position of a point will, ever after, be content with our relative knowledge."

4. In order to have a clear view of what this means, it will be necessary to go into further details.

We are accustomed to talk of a point of space, marking position only. This position, however, is that of a small body, which we imagine to be there and the position itself is defined by means of distances from three planes which are called planes of reference. We are thus led to the cognisance of Space, really through material objects, though there is little justification for the view held by Descartes that Space is the only form of substance and all existing things are affections of Space or the still more subtle conclusion of his that if the matter within a vessel could be entirely removed, the space within the vessel would no longer exist. On the other hand, Leibnitz defines space as the "order of possible co-existing phenomena." In fact, whenever we attempt to go behind the postulate that the notion of Space is primordial, we get lost in the mazes of metaphysical subtlety.

5. Although, however, a cognition of space is necessarily associated with its material contents, we are able to make abstraction of the latter and to regard space, only in its geometrical aspects, as possessing only the property of extension in three directions. And Physical Science is concerned with the measurement of this extension, although it will be perhaps almost impossible to detail further as to what extension is in itself. Now, for the purposes of this measurement, we have to provide ourselves with a measuring rod and fix upon the planes of reference from which directional extensions or distances are to be measured, called a Frame of Reference and it will be noticed that this frame of reference—for example, the two walls and the floor of a room—may be regarded as only spatial and not material, while the

measuring rod itself may be regarded as having extension in one direction only. In both cases therefore, there is a double abstraction—both of their materiality and extensions in other directions. There are other processes of abstraction, which we shall have consider in due course. In the meantime, it is obvious that the metrical properties of space, obtained by comparison with the measuring rod are, to that extent, valid. It is abundantly clear however that space thus conceived is relative—not merely in the sense that it is relative to our own power of perception and comparison, but also in the sense that it is relative to a given Frame of Reference.

6. It is also relative in a third sense, which in its simpler aspect is also matter of common knowledge and to which a brief reference is necessary. To a traveller in a Railway carriage, a point inside appears to occupy the same position in space, or the same position relative to any chosen frame of reference (say, any two walls and the floor of the carriage, which in this case is in motion). If we are unconscious of our motion, it would be impossible to realise that the given point does not occupy, as a matter of fact, the same point of space. In other words, the motion of the observer and the frame of reference complicate the phenomena and require to be taken into account in any reasonably accurate analysis of the result.

7. Before we proceed to this discussion, it is necessary to note that we may analyse the process associated with the concept of space, further. The conception developed so far is statical, but only because we have unconsciously learned to overlook one essential feature connected with it. For it really involves the idea of transference—of motion, i.e. of change of position in time, so that the idea of time is intimately associated with that of space. And it will be seen that time enters so fundamentally into the measurement of space, that it may be regarded as a characteristic property of space, in the same way as its extension, proper, in three directions. By an obvious extension of meaning of the word dimensions therefore, we may regard time as a dimension of space and since all physical phenomena are conditioned by space and time, we may describe these as occurring in four dimensional space. Such a statement may, at first sight, seem to be misleading but if we remember what it really means, there is no reason to be apprehensive. Analytically, it amounts to the statement that if ϕ is any physical quantity, conditioned by time and space or an event

$$\phi = f(x, y, z, t),$$

and that $\frac{d\phi}{dt}$ is never zero.

A further and stronger justification arises from the fact that all physical investigations ultimately lead to a differential equation of the form,

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=c^{-2}\,\frac{d^2\phi}{dt^2},$$

so that putting

$$\frac{1}{ic}\cdot\frac{d}{dt}=\frac{d}{d\omega}\,,$$

we get an equation in which any characteristic difference that may physically exist between x, y, z and w is seen to be obliterated.

It is easy to see also that

$$\frac{d\phi}{dx}\,dx+\frac{d\phi}{dy}\,dy+\frac{d\phi}{dz}\,dz+\frac{d\phi}{dt}\,dt=0$$

is a line ('a world line') representing the path of an event giving the history of an event corresponding to a particular series of changes of x, y, z and t. Now, this concept of time as we have seen is primordial. It is, however, relative in the sense that it was probably originally based on a recognition of an order of sequence in our own consciousness. By making abstraction of this, we arrive at the conception of "absolute, true and mathematical" time, as conceived by Newton, which is flowing at a constant rate, unaffected by the speed or slowness of motion of material things the independent variable in all investigations of Newtonian Dynamics Such a conception presents no practical difficulty, for any required interval, in terms of this absolute time, corresponding to any phenomena, that we deal with in science is *taken* to be equal to the interval measured in terms of what, by antithesis, may be called relative time. All that is necessary is that we should be able to estimate by some mode of measurement an interval of time, relative or absolute.

8. As this is a point that bears directly on our discussion, it is necessary to note the remarkable fact that though time (absolute or relative) is a quantity which we are unable to define but can only be conscious of, we are able to measure it. This, it should be remembered is a process we are continually resorting to in ordinary life. The quantity of electricity, for instance, that we consume from day to day

is being measured and charged for, though no one, certainly none even among those whose duty it is to measure and charge for it can tell what electricity is. Similarly, when a grocer measures out a quantity of sugar he follows a strictly scientific procedure, though he could not tell what sugar was. For the purposes of measurement, all that is necessary is that we should fix on a standard—a standard pound, for instance, in the case of the sugar, which must be an invariable quantity, and should know how to compare the quantity to be measured with the standard, on the assumption that neither the quantity to be measured nor the standard changes during the process of comparison—the measurement having reference to certain properties common to the two. The process in the case of the sugar is, as we all know, that of weighing and it is readily seen, that if we had any reasons to suppose that weighing is affected by any cause or circumstance, that cause or circumstance will vitiate the process and must be taken account of in rectifying it. Now it is known, that the weight of a body depends on its position on the surface of the earth, so that the weight as a measure is in that sense only relative; nor would it have been possible for us to detect its variation, if we had only marked weights with which to carry out the measurement, as marked weights change in the same ratio as the stuff to be weighed; in order to detect this variation, therefore, it is necessary to devise a process of measurement (by means of a spring balance, for instance), which is independent of the force of gravity.

9. We, thus, observe that we habitually measure quantities, of the intimate nature of which we may be profoundly ignorant, provided we can fix on a unit of the same nature, of invariable magnitude and can compare it with the quantity itself, under circumstances which do not introduce an extraneous error. Time is dealt with by the Astronomer and the Physicist in this way. The unit of time chosen is the sidereal day, or the period of the earth's rotation about its axis, which for most purposes may be taken to be a constant interval and measurement of any other interval (relative or absolute) in terms of this unit is to that extent valid. But relativity of time extends beyond this in ordinary life. For the practical unit of time is different from the sidereal day—being dependent on the motion of the earth and the observer round the sun. This unit is called the mean solar day. It is moreover common practice, to use different kinds of time, e.g. the local time, the local standard time and a general standard time,

(such as Greenwich or Paris time). This relativity may be said to arise from the position of the observer, or the frame of reference to which that position is referred. Thus, if t_s=measure of a sidereal interval and t_m=that of a mean solar day,

$$t_s = (1 + \omega)\, t_m$$

where ω = the mean (daily) angular *velocity* of the earth round the sun.

Again, if T = local mean time. T_G = Greenwich meantime.

$$T = T_G + \frac{}{\omega}.$$

where θ = longitude of the observer measured westward.

It will be seen, presently, that the form of this relation is similar to that necessitated in the theory of relativity. Another interesting point in the relativity of time, is also a matter of common knowledge, namely the fact of dependence of time on the motion of the observer. In coming from the West, for instance, it is necessary to put on the clock hand as one moves further and further away, the opposite being the case, in going West. Moreover, two ships starting from the same point and going in opposite directions will, when they meet, find a difference of 24 hours in their reckoning of time. The relativity of time is, thus, found to depend not merely on the unit chosen, but also on the place of the observer and his motion. There are other factors also associated with the new point of view, which we shall consider in due course.

10. When we have learnt to measure space and time, change of position in time or motion can be determined And the question naturally arises—what is it that moves ? In ultimate resort, all movements probably involve a transference of a 'material' something, but in some cases, at least, we are aware of a transference but not that of anything which can be described as material. Moreover, a transference may be a bodily transference or one caused by vibratory or rotational movement. When sound is emitted by a sounding body and is received by the ear, the transference involved is not a bodily transference of material particles but a transference of the vibratory motion of the sounding body to the ear, and it is well known that the intervening space is filled with a material medium—the air for example which takes up the vibration and transfers it at a rate, depending on its elastic properties. Further, the rate of transfer will be differ-

ent, if the sounding body or the ear or the medium or all or some of them is in motion. It is well known, for instance, that the pitch of the whistle of an advancing train is raised and a similar effect is produced when the wind is blowing away from the observer.

11. In this case, the medium (air, for instance), is a direct object of sense, whose properties are subjects of laboratory experiment, so that the mechanism of sound-propagation and the manner in which motion affects it are capable of adequate experimental test. When, however, we proceed to examine the mechanism of light-propagation in the same way, we are confronted with formidable difficulties. For the mechanism of the propagation of light cannot be subjected to direct experimental test, as in the case of sound, since the propagation takes place through intersteller space, which certainly cannot be described as a material medium in the usual sense.

The property of space as a vehicle of light is a new property, which has long engaged the attention of natural philosophers. Records of their speculations are available from the time of Aristotle and the Nyyaya Philosophy, downwards. Since the theory of corpuscles shot off from luminous bodies to produce the sensation of sight has been found incapable of giving an adequate account of the various phenomena of light, space, regarded as the luminiferous medium has been held to be a plenum, filled with a subtle fluid of some kind, which is capable of taking up and transferring the vibratory motion, of which light, for very cogent reasons, must be held to consist. This postulated fluid medium is the so-called ether of space. Adequate reasons are also forthcoming for the view that this medium is a receptacle of energy, of various kinds—not merely the energy of light and electrical energy which is identical with light energy but perhaps also that of gravitation.

12. Since wireless messages began to be sent through space, it brought home to the lay mind, as much as to the scientist that the ether of space, as a transmitter of vibratory motion of which these messages consist is identical with the medium which transmits light-signals to the observer from the remotest stars. As these messages are produced by electrical means and passage of electricity is associated with magnetic effects, the medium is also, fittingly, called the electromagnetic field.

13. Now, what is the effect of motion, on these phenomena

of which the ether of space is the seat? In particular, in what way are the light signals from distant stars affected by motion?

Imagine a frame of reference situated in the ether to which the motion of transfer of light from a star to the observer may be referred. Let the frame be such that the sun and the stars may be regarded (as a first approximation)—as at rest with reference to it. Then the motion of the earth with respect to such a frame will be that relative to the sun. In this case, the direction in which light is received by the observer, moving with the earth will be inclined to the actual direction, just as, to a person moving forwards, rain drops, falling vertically appear to come in a slanting direction. The result is that all stars would appear to move in similar closed curves, parallel to the earth's path, completing a cycle, in the time that the observer takes to complete its path round the sun, i.e. a year. And extended observations have shown that this is actually the case.

14. The meaning of this agreement between the observed displacement and the calculated amount might at first sight appear to be simply that our postulated frame of reference is fixed in space. But it appears that this is not justifiable. For this effect is not related to the motion of the medium, but only to the motion of the observer *relative* to the medium supposed to be at rest.

Direct experiment has shown, however, that when a material medium is in motion, the velocity of light through such a medium is thereby increased, the effective increase being less than the actual velocity of such a medium by a small fraction. This remarkable result (which can be theoretically deduced on certain assumptions) leads to a serious difficulty of an unforeseen kind. For, if we pass light through a tube full of air, oriented in the direction of the earth's motion or perpendicular to that direction, no detectable difference in its velocity is observable. This is the celebrated null-effect of Michel-and Morson ley's experiment. But the motion of the material medium (the air in the tube) in the first case is that of the earth relative to the ether, as measured in the direction of light propagation, whereas in the second case, it is *nil*, measured in the same way. Since then, no difference is detectable, the conclusion is either that the motion of the earth relative to the ether is *nil*—which is the exact opposite of the former conclusion (art. 13) or that the tube contracts, automatically when oriented in the direction of the earth's motion. If the latter is accepted, it is somewhat remarkable, that no experiment has hitherto been able to detect this contraction. It was, in fact, reasonable to suppose that this contraction would be associated with corresponding, electrical, mechanical or optical effects but none such, has so far been actually detected We must conclude therefore, that this contraction—if it must be admitted to exist (and it has to be admitted, in order to explain the null effect above referred to) is compensated for by effects, as deep-seated as the electrical and other properties of bodies. But since these properties have been analysed into those of their ultra-atomic structure, it is clear that these contractions must be held to affect that structure in a manner, such as precludes the effect being observed by any known device.

15. We may, in fact, regard the frame of reference to move with any arbitrary velocity that we choose. This would seem at first sight to lead to the conclusion that we are precluded from discovering any physical laws, whatsoever. We are, however, saved from this intellectual barrenness, by the new principle of relativity. This is based on Einstine's postulate which admits that the laws of physical phenomena are the same whether these phenomena are referred to any frame of reference or to any other frame moving uniformly with respect to it with any arbitrary velocity whatever. If, however, these phenomena as conditioned by time and space are to be capable of scientific treatment, there must be some connecting link between any two frames of references, moving with an arbitrary velocity, with respect to each other. This is supplied by the additional postulate that velocity of light is the same, whatever be the frame of reference, whether the source be at rest or in motion.

16. When this additional postulate is introduced, it is found that the times in the two frames of reference are related to each other, not only in the same way as the local time and standard time of our ordinary (astronomical) modes of reckoning but that the units have to be different (cf. sideral and mean solar days) being related to each other as the units of length in the two frames. In other words the relation depends on the relative velocities of the observers. This is an additional principle of relativity, to which time-reckoning is subject, new to science—at any rate in the form in which it has been enunciated in the modern theory of Relativity.

In fact, if this contraction is independent of the material used and is automatic, it necessarily follows that it will not be possible to detect this contraction except as a matter of indirect deduction, as set forth here. We have already seen that gravity as an universal property of matter is not detectable directly, as all bodies are equally affected, except by a special device. Here, the effect is more deep-seated still for we fail to detect it by any device whatever. And this stands to reason, as it depends on the relative velocity of the earth to the all-pervading ether. Admitting, then, the existence of this contraction, we have the remarkable result that the length of a body depends on the velocity of the body, relative to the frame of reference, which we have assumed and that this relative velocity is a quantity that cannot be directly measured.

17. The formulae of transformation as we pass from one frame of reference to any other frame of reference, moving with any arbitrary velocity whatever in relation to the other have been derived in various ways. I have investigated them on the single postulate that the electric charge is an invariant for all such transformation.

In a paper in the *Phil. Mag.* (Apr. 1915) I have shown that Maxwell's eqn.

$$\frac{\partial f}{\partial x}+\frac{\partial t}{\partial y}+\frac{\partial f}{\partial z}=\rho$$

can be interpreted in terms of the electron theory by supposing

$$\frac{\partial A}{\partial x}+ - = - \rho,$$

where I (A,B,C) is the coeff. of electrification giving rise to electric moment

$$\partial M = Id\sigma \ (d\sigma = \text{element of volume})$$

And the electron theory of dispersion gives

$$C = C_0 e^{ip\,(nz\sqrt{k_0}-t)}. \ \&c.$$

where $\frac{1}{\sqrt{k_0}}$ = velocity of light in free space, where the medium is fixed.

That is, $$C = C_0 e^{pi\,(nz\sqrt{k_0}-t)}. \ \&c. \quad (1)$$

rel. to the moving observer, moving with the medium (vel. ζ)

while, $$C' = C_0 e^{cp'\,(n'z'\sqrt{k_0}-t)}. \ \&c. \quad (2)$$

relative to the fixed observer

Also, $$z = z' - \zeta t.$$

where, (1) is the solution of equations of the type,

$$\ddot{C} + p_0{}^2 C = c_0 h.$$

$$\ddot{A} + p_0{}^2 A = a_0 f$$

and

$$\left.\begin{aligned} \ddot{f}'' + \ddot{A}'' - \frac{1}{k_0}\frac{\partial^2 f}{\delta z^2} = 0 \\ \ddot{h} + \ddot{c} - \frac{1}{k_0}\frac{\partial^2 h}{\partial z^2} = 0 \end{aligned}\right\}$$

and (2) is the solution the same eqn. where

$$\frac{d}{dt} \text{ is changed into } \left(\frac{d}{dt} + \zeta\frac{\partial}{\partial z}\right)$$

whenever it operates on C and there is an additional term due to motion of electrons

$$\frac{k_0}{4\pi}(-\zeta\beta),$$

where β is the magnetic force due to this motion.

In this way it can be shown (Poincaré, *Electricité et Optique*) that up to the ***first order***

$$\frac{1}{n'} = \frac{1}{n} + \zeta\sqrt{k_0}\left(1 - \frac{1}{n^2}\right).$$

But $\frac{C}{z} =$ $\frac{\partial A}{\partial x} = \frac{\partial A}{\partial x'}\sqrt{\frac{c^2-\zeta^2}{c^2}}$,

$$\frac{\partial B}{dy} = \frac{\partial B}{dy'}\sqrt{\frac{c^2-\zeta^2}{c^2}}, \text{ from the geometry of relative motion.}$$

and $$\rho\, dx\, dy\, dz = \rho' dx' dy' dz',$$

we must have $\frac{\partial C}{\partial z} = \frac{1}{\lambda}\frac{\partial C}{\partial z'}$, so that $\lambda^{-1} = \sqrt{1 - \frac{\zeta^2}{c^2}}$.

This shows that the above equations (1) and (2) require further modification, viz., if

$$C = C_0\, e^{ip\,(nz\sqrt{k_0} - t)}$$

relative to the moving observer in the moving medium,

we must put $$C = C_0\, e^{ip'\,(n'z'\sqrt{k_0} - t')}$$

relative to the fixed observer

and ζ being the velocity of medium referred to the second observer

$$z = \lambda\,(z' - \zeta t')$$

$$t = \mu\,(t' - \nu z'), \text{ where } \mu, \nu \text{ have to be determined}$$

therefore, $$C = C_0\, e^{ip\,[(n\lambda\,(z' - \zeta t')\sqrt{k_0} - \mu\,(t' - \nu z')}$$

Now, comparing coeff. of z', and t',

we have
$$p\,(n\lambda\sqrt{k_0} + \mu\nu) = p'n'\sqrt{k_0} \quad (3)$$
$$p\,(n\lambda\zeta\sqrt{k_0} + \mu) = p' \quad (4)$$

Hence, from (3) and (4)
$$n'\sqrt{k_0} = \frac{n\lambda\sqrt{k_0} + \mu\nu}{n\lambda\zeta\sqrt{k_0} + \mu}.$$

i.e.
$$u' = \frac{\lambda\frac{\zeta}{u} + \mu}{\frac{\lambda}{u} + \mu\nu} \text{ where } \frac{1}{n'\sqrt{k}} = u' \text{ \&c.}$$

$$u' = u + \zeta\left(1 - \frac{u^2}{c^2}\right)$$

up to the *first order*

Hence,
$$i.e.\ \lambda = \mu \text{ and } u\nu = \lambda\zeta\frac{u}{c^2}.$$
$$\nu = \frac{\zeta^2}{c^2}.$$

We have accordingly $z = \lambda\,(z' - \zeta t')$, (5)

where
$$\lambda^{-1} = \sqrt{1 - \frac{\zeta^2}{c^2}} \text{ and } t' = \lambda\left(t - \frac{z'\zeta}{c^2}\right), \quad (6)$$

18 The form of the 2nd expression is, as we have already indicated similar to the relation between local time and standard time, while λ introduces a change in the unit. The eqn. (5) is an extension of the usual formula of transformation
$$z = z' - \zeta t'.$$
which is characteristic of the new theory.

The generalizations open out a wider purview to the physical enquirer in a manner, which is not, however, altogether, new to science.

19. As a simple example it may be noticed that the science of quaternions was developed by a process of extension, as soon as it was recognised that the *commutation law* was only of limited application.

20. The result of the modern theory is remarkable, nevertheless, but the interest is entirely scientific, so that there is nothing in it that is likely to affect our everyday experience. As this is a point, which has been much misunderstood, it would be worth while pausing for a moment to consider in detail its practical significance.

21. According to old (Newtonian) dynamics, a body does not change its shape and size because of its motion. On the present view,

it does. But to what extent? In the first place, as every physical property of a body and that of all bodies change simultaneously, it would be impossible to detect it. And even if we could detect it, it will be only appreciable, if the velocity of the body relative to the observer is comparable to the velocity of light, which is 186,000 miles per second. Similar remark applies to the change in regard to time, so that it appears that the man in the street, as well as men in higher places may well go about their usual avocations without fear and trembling, for any change that the "new theory of the universe," (to quote the popular description of Einstein mechanics), involves.

22. We may regard the interesting feature of the present view in another manner. Suppose we assume the earth's velocity relative to the ether to be as much as 161,000 miles per second in a vertical direction. Then* a rod six feet long, when horizontal contracts to 3 feet when placed vertically. But the standard yard-measure will change in the same way, so that it will not be possible to notice the change. Nor will any change be observable, as the rod is rotated from the horizontal to the vertical position, if we admit that the image in the retina changes *pari-passu*, which is apparently the case. We may apply other tests, electrical, optical, etc. But they all fail, as we have already seen. Thus one is disposed to reiterate the conclusion that the practical position is unchanged, in spite of Einstien's theory. What is changed, however, is the point of view—the aspect of things in a broader sense than heretofore.

23. In dealing with motion, so long as we confine ourselves to motion at a given position on the earth, we regard the axes as fixed in space, if they are fixed on the earth. The results obtained are of course valid, but only to that extent. This is altered, if we have to compare motions at two different places on the surface of the earth; we then must have recourse to two sets of axes, moving relative to each other in a known manner. When however, we proceed to celestial motion, generally, a frame of reference independent of the earth's motion is required and the plane of the earth's orbit or the "invariable plane" is taken as the plane of reference and a certain initial epoch, convenient to the Astronomer, as the point from which time is measured. We have, thus, as already indicated, various degrees of rela-

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* Eddingtion.

tivity in ordinary mechanics, but it will be seen that the peculiar kind of relativity that we are now considering is of an extended scope and significance.

We may analyse this aspect further. When a rod is set moving uniformly, we say, according to the present view, that it contracts, but the contraction is only a way of describing the new spatial relation that comes to subsist between the rod and the observer, the contraction being only relative to the particular observer--being different for different observers. If, then, the spatial relations are different for different observers, we reach the remarkable conclusion that each observer carries his own (four dimensional) space with him, *so that the distinction between the perceptual and the conceptual on the one hand and between the subjective and the objective on the other seem to tend to be obliterated.

24. Prof. Karl Pearson speaks of "a cheap and unfortunately common form of emotional science which revels in contrasting the infinites of space with the finite capacities of man.'' He argues that "the space of our perception, the space in which we discriminate phenomena, is exactly commensurable with the contents of that finite capacity, which we term our perceptive faculty, so that the only infinite space, we know of is a product of our own reasoning faculty.'' ''The mystery of space,'' according to Karl Pearson, "whether it be the finite space of perception or the infinite space of conception, lies in and not outside each human consciousness.'' The theory of relativity suggests, however, that space possesses properties which are neither entirely perceptual nor entirely conceptual but which partakes of both at the same time. In the same way, the old formula that the property of a body, extension for instance, is either in the body or in the consciousness of the observer is seen to be inadequate. For, after all, it appears that it depends on both in a more emphatic sense than, we have been used to, so far. The other consequences of such a scheme are naturally far-reaching, but only for the purposes of scientific exactitude. Thus, just as the size and shape of a body are dependent on the velocity of the body, its mass also is found to be dependent on it. But it is dependent in so minute a measure that we need not

* The illustration of the rainbow has been used by Jeans, which seems to be very

trouble about it (until, perhaps, we have occasion to exchange commodities with Jupiter).

25. Again, for establishing communication between different systems, the sole constant involved is the velocity of light in vacuo. With regard to this point, it will be remembered that practically all our observations are based on an optical method and all our standards depend on this velocity. Since, therefore, any uncertainty in this would introduce a complete uncertainty into the entire range of our experience, the postulate of constant velocity of light is *apriori* justified.

26. The justification of a theory is best measured by its power to explain outstanding problems and, accordingly, an attempt has been made to apply the theory of Relativity or a modification of it to explain the most celebrated of these, viz. the intimate nature and Law of Gravitation. This has led to a 'generalised' theory of Relativity as distinguished from the 'restricted' theory, which we have been considering, so far.

For it should not be forgotten that the so-called gravitational law itself requires explanation. There is, in fact, nothing fundamental in the law that bodies attract each other, and that the attraction varies inversely as the *square* of the *distance*, and as the product of their *masses*. It does really nothing more than supply a hypothesis—an artificial hypothesis at that—on which the motion of the solar system can be explained in a tentative manner, as a first approximation. The intimate mechanism of the processes which actually yield the result summarised under the law have accordingly awaited discovery since the time of Newton, who himself attempted it. And now that masses are found to depend on the velocities of bodies—and space and time are held to have new attributes, the law has come naturally to be under careful scrutiny, on the new principle of relativity. This has been done by Einstein himself and it appears that a modification of the Newtonian theory is called for. And it is moreover found that when this modified theory is applied to specific problems, the results are singularly satisfactory.

27. It is difficult, if not impossible to give an account of the manner in which the so-called Law of Gravitation has been analysed and modified, on the Relativity Theory. Without attempting to reproduce the somewhat complicated mathematical analysis, I propose only to indicate the general line of thought.

28. When a top is set spinning, it can be made to stand but only while spinning—not otherwise. This may be taken as an illustration of a 'field of force,' apparently brought into play—the so-called centrifugal force—as the effect of a change of a frame of reference in the four demensional, space of the restricted theory. For in this case, when the top is not spinning, we may take for the frame of reference the axis of the top and two lines at right angles to each other (provisionally fixed in space).

This field of force, then, may be described, as arising from a change in spatial relations in the spinning top from those that obtain in the stationary top. Now this change is independent of the nature of the body, and is dependent only on a geometrical deformation—with reference to the room, as a frame of reference. When it is spinning, this frame changes its position continuously. Similarly, when a stone is whirled round, at the end of a rope held in the hand, the pull at the hand may be said to evidence the play of a field of force, which, although we may otherwise explain it, undoubtedly represents a real property of Time and Space. These illustrations are necessarily crude, but they enable us to make a mental picture of the principle laid down by Einstein and called by him the Principle of Equivalence:—*viz.* a gravitational field of force is exactly equivalent to a field of force introduced by a transformation of the co-ordinates of reference, although there may be no means of distinguishing between them experimentally; only, their equivalence is to extend merely to the regions, over which the transformation is possible.

29. An illustration of a different kind from those already given though still somewhat crude, may also be given. Going up in a lift, one feels the pressure between the feet and the lift increasing with increased acceleration. When the lift descends, the pressure decreases, till, if the lift falls freely, the pressure becomes *nil.* With reference to the frame of reference rigidly fixed to the lift, which is moving with an acceleration, the forces operating (in this case the pressure) acquire values depending on the motion of the frame of reference. The resulting field of force is entirely due to transformation of axes from those fixed to the earth to those fixed to the lift. In the particular case, in which the effects of the gravitational field are neutralised, the principle of equivalence tells us, all other effects will be neutralised also.

30. In fact, the motion discussed in Rigid Dynamics, of bodies

under no forces supplies ample illustration of forces arising from changes in spatial relations. And, since such changes are the fundamental postulates of the theory of relativity, it was natural to inquire as Einstien did, if suitable types of these changes in *four-dimensional* spatial relations could account for gravitation* also. This has led to the Relativity theory of gravitation and to a modification of the Newtonian principle. The inquiry has, in fact, led to the result that the motion of bodies in a gravitational field may be explained, without invoking a gravitational force following Newtonian law—which is now known to be only a first approximation and as arising from the curvature of space, imposed by bodies (such as the earth or the sun) to which the field was supposed to be due—the actual path being the shortest in the new four-dimensional space of Einstein Kinematics.

Such a hypothesis can only be justified by its applications to actual cases. It will be necessary therefore to refer to a few of these.

One of the most successful of these applications is to a well-known problem in the planetary theory, namely the celebrated discrepancy between the observed period of rotation of the orbit of Mercury (574 seconds per century) and the calculated amount on the Newtonian theory of perturbations, due to the action of the other planets (about 532 seconds). This can be completely accounted for, on the Theory of Relativity. The planet in pursuing the shortest course through the curved four-dimensional space round the sun describes the path as it is known to do, rather than the path indicated by Newtonian theory. And although it has been argued on the other side that a suitable modification of the Newtonian Theory is competent to yield the same result, it appears that a final pronouncement should be in favour of Einstein's Theory. But a more striking confirmation of the new theory of gravitation based on the Principle of Relativity is now forthcoming. Einstein had predicted that rays of light would suffer deviation in a gravitational field and that if this is to be the case, rays from stars which have to pass through regions in the neighbourhood, of the sun in reaching the observer should suffer a known deviation

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* A further justification for this line of argument is supplied by the formulae of transformation connecting, x', y', z' and t' with x, y, z, t. For they are seen to be the special case of the linear transformation

$$x' = \alpha x + \beta y + \gamma z + \delta t, \text{ etc.}$$
$$t' = \alpha' x + \beta' y + \gamma' z + \delta' t.$$

on account of the sun's gravitational field *i.e.* in following the shortest path on the curved space round the sun. In fact for a star seen close to the limb of the sun the light ray should experience a total deflection of 1·774″. This prediction of theory has been amply confirmed from observations at the last total eclipse of the sun. Even the slight discrepancy that was noticed at the time has since been held to be due to instrumental distortion.

31. The Theory of Relativity must, therefore, be held to give an insight into the nature of Space and Time, which is new to Science. If this be so, it suggests a difficulty which should be carefully considered. For the Theory of Relativity leads to the conclusion that any frames of Reference are valid, no matter how they are moving relatively to each other, provided the unique condition as to the constancy of the velocity of light is satisfied, with reference to them. If one of these frames of reference be situated in the ethereal medium—that all pervading medium through which light and electric disturbance are propagated (the so-called electro-magnetic field), this apparently leads to the conclusion, that the ether may be supposed to be moving with any arbitrary velocity whatsoever, or, since each observer carries his own space with him, each observer will have his own "ether." This has sometimes been held to mean that there is no such medium, for, i t has been argued, that such a medium, if it existed must necessarily e inert. This view, however, does not seem to be, altogether, justified.

For various lines of argument seem to point to the conclusion that the ethereal medium, through which light is propagated cannot be held, altogether to be inert or immobile. Cogent reasons can be advanced for the view that there is intrinsic—if concealed motion in the medium. If, further, the same medium takes part in the propagation of gravitation, the motion of the medium may be arbitrary in the sense that the actual energy of the gravitational field cannot be specified. It is, morever, admitted that absolute unit of electricity is a quantity which is independent of a frame of reference. But the most satisfactory view regarding the intimate nature of this quantity that has, so far, been put forward is that it is of the nature of a singularity in an all pervading medium arising from motion or disturbance of some kind. If that be so, we seem to come back to the postulate of a medium. It has indeed actually been shown by Weyl that the curvature of generalised space postulated by Einstein to explain gravitation is due to a distortion which is a particular case of the most general distortion of which such a space is capable, and which will explain electrosnagretic phenomena as well.

32. In fact, all our so-called explanations of Physical phenomena reduce themselves to the preparing of models that shall represent the actual as nearly as possible, but can never be the actual itself. It does not appear therefore that the theory of Relativity will dispose of the physical existence of the ethereal model until a better one can be found which shall explain the intimate nature of the various concepts of modern physics, corpuscles and material particles, electric charge and magentic force, gross matter and gravitation in one comprehensive scheme.

A STUDY OF FATIGUE AND ENDURANCE

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Materials for this paper were gathered in course of the Health Examination of the students of the Scottish Churches College, Calcutta, in April last. The immediate aim of the examiner was to make a comparative study of physical fatigue and endurance in certain standard conditions of work. Naturally, with the time at his disposal, he cared more for convenience than for scientific precision. The materials are not, therefore, what an exact scientist would like them to be. They are under the influence of a large number of chance and systematic errors, many of which can with difficulty be controlled. Yet they are useful in their own ways for the scientific study of fatigue. Allowances may be made for known errors; and by mathematical treatments of the results, the influence of some errors may be minimised, while that of others may at least be indicated.

In the present paper, I shall discuss certain preliminary results which I cannot presume to present as final in any sense, but which would give us some idea of our problem. The number of measurements upon which the results of this paper are based is small and hence much of what has been written in it may have to be corrected in future. I wish to discuss the problem more fully in future when a large number of measurements are obtained.

§ I.

Method.

The degree of fatigue that a living tissue or organism has undergone in consequence of continuous work for a length of time under certain conditions is measured by the amount of its loss in efficiency for that time. Provided the conditions of work remain the same and the organ or organism continues to do the same work, we can compute the loss in efficiency by comparing its rate of work at any particular moment of fatigue with that at the beginning or any other period of efficiency. For this we require a series of records of work done in a certain period of time under certain uniform conditions.

Mosso's Ergograph is ordinarily used. Though it has the advantage of studying fatigue of a single muscle and thus under simple conditions, it is a rather complex apparatus and cannot conveniently used for securing materials from a large number of subjects. We chose therefore, the dynamometer (Smedley type). It is convenient to handle and we hoped that, as in the dynamometer test the subject is required to work at *his* best, without being restricted to a particular resistance, it would indicate more correctly, than the Ergograph, the general working power of the subject. Also an ergogram with a variable load, according to certain, authorities, "gives a truer picture of the course of muscular fatigue", than an ergogram with a constant load. And as the curve of work from a series of grips, as well as from a single continuous grip, with the dynamometer corresponds to this ergoram with a variable load, we thought it to be better suited to our purpose than the cumbrous Ergograph.

Records with the Dynamometer can be secured by two different methods. In the first method, called the method of continuous contraction, records are taken by the examiner at the interval of 3 or 4 secs. while the subject continues pulling the inner stirrup of the Dynamometer with his maximal effort, say, for 1 mt. In the second method, called the method of separate contractions, the subject pulls with his maximal effort only at the interval of 3 or 4 secs. and the examiner takes record of every maximal grip.

The second method was adopted here. I tried the two methods separately upon the same individual, at the interval of half an hour. The two series of records were substantially identical. Further, the second method is more convenient to apply. In it chance of error due to the examiner is small as less strain is put upon him. In the third place, in the first method the subject has to maintain his maximal attention for more than 1 mt. But attention is naturally rhythmic and our maximal efforts are also rhythmic in character. The second method follows this natural tendency of attention.

We have followed the instructions given in Whipple's *Manual of Mental and Physical Tests*, Vol. I, except that they were slightly modified.

Care was taken that the adjustment of the inner stirrup fitted the hand of the subject. The recording pointer on the dial was moved away to the right. The subject and the examiner sat on the opposite sides of a low table, the subject holding the dynamometer before him so that the record could be easily taken by the examiner. The inner stirrup pressed against the second row of phalanges. Whipple instructs that a metronome is to be set beating once per sec., the examiner is to call "Now" at every fourth beat of the metronome whereupon the subject is to pull as forcibly as he can. We have substituted the metronome by a second's pendulum set upon the table just before the eyes of the subject. The subject is trained at first to count the oscillations and how at every third movement of the bob towards him he is to grip with is maximal effort. He is not to look to the recording of the grips but to keep himself occupied with the counting of the oscillations. This arrangement lessens the labour of the examiner and also serves as a condition for testing the "Co-ordinating" ability of the subject. The sight of the oscillating bob often induces a rhythmic Kinaesthesis in the subject and this, no doubt, helps him in keeping up the attention. After the subject has learnt to count the oscillations and understood how to gril at every third oscillation with his maximal effort, the examiner calls "Ready" and the subject begins pulling every third second with his maximal effort till he was told to stop at the end of 1 mt. The examiner in the meantime takes note of records. We chose the interval of 3 secs. instead of 4, as it induces fatigue more readily and gives large number of records within the same time. The right hand was first examined and then the left.

§ 2.

Treatment of Results.

A sample record:

Table 1.

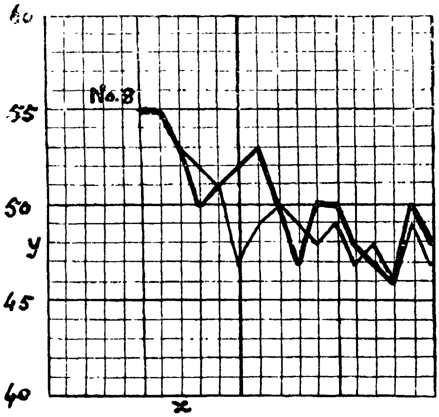
Name :—

Right hand.				Left hand.			
55	51	47	47	55	51	49	48
55	52	50	46	55	47	48	46
53	53	50	50	53	49	49	49
50	50	48	48	52	50	47	47

(The figures indicate strength of grip in k.g.)

The Fatigue index is obtained by the formula $x = \frac{r_1 - r_2}{r_1}$, where x = the required fatigue index, r_1 = the mean of the first four readings, and $r_2 =$ the mean of the last four readings. In the record given above, $r_1 = 53{\cdot}25$ and $r_2 = 47{\cdot}75$, therefore, $x = {\cdot}103 \ldots$ for the right hand. (Though we have usually taken 20 readings in 1 mt, each at the interval of 3 secs., we have computed the fatigue index from 16 readings in order that we may compare our results with those of western investigators.) The mean of the first four readings would fairly represent the working capacity of the individual at the time of the examination in its best phase of efficiency. The mean of the last four readings would similarly represent the same capacity impaired by fatigue. The ratio which the loss of the working capacity would bear to the original capacity gives an index of fatigue for the individual and this index can be trusted if we can admit that the subject has gripped with his maximal effort everytime and no irrelevant factor has disturbed the record. In certain cases the phase of efficiency is not reached in the first group of four readings, but only in the second, so that the mean of the second four readings is higher than that of the first four. In such cases maximal efforts are exerted only gradually and hence the index is calculated by the same formula from the last 16 readings instead of from the first 16 readings, as ordinarily.

In our measurement of fatigue we need not take into consideration the intermediate readings between the first four and the last four. But they are valuable in indicating the course of fatigue or in other words, they show whether fatigue has set in gradually or abruptly, etc. If the readings in the above record be plotted in a graph. We can get an endurance curve, like that shown here.



No. 1.

Along the abscissa x are plotted the time intervals of 3 secs. each and along the ordinates y are plotted the grip readings in k.g. No. 8 indicates the serial number of the subject. The black heavy line indicates for the right hand and the red line for the left. Endurance curves are made up of all sorts of zigzag lines, so that if we are to classify them into types we should first smooth them. The small rises and falls that are present in most of the endurance curves are due to: (1) chance errors arising from holding the instrument in a slightly wrong position; (2) momentary flashes of enthusiasm and depressions of the subjects; (3) parallax error, and also to other unknown causes. The smoothing would to some extent minimise the influence of these slight errors and yet represent the general tendency of the curve.

I adopted the ordinary simple method of smoothing, viz. by drawing a smooth line, by the eye, through the curve so that it passes through the largest possible number of the plotted points. For the determination of the different types of endurance curves, I used the method of plotting the "moving" averages.

Binet and Vaschide divided their subjects into three groups: (1) strong; (2) medium; (3) weak, according to their maximum grips, in order to examine the manner of distribution of endurance types in the several groups. I also divided our subjects into the same three groups and worked out the average fatigue index of each of them.

§ 3.

Results.

(Altogether 100 college students ranging in age from 17 to 23 were examined. Of 100 records 10 were rejected as unsatisfactory. In the calculation of certain results age differences were not taken into account. Other results were limited to particular age groups.)

Table 2.

Fatigue Index of 30 *Students, all of age* 20.

Order.	Right Hand.				Left Hand.				
	No.	Standing.	*d.*	*d*².	No.	Standing.	*d.*	*d*².	*xy.*
1	57	·03	—·12	·0144	43	·07	—·12	·0144	— 0024
2	55	·05	—·10	·0100	98	·07	—·12	·0144	·0020
3	101	·07	—·08	0064	15	·08	—·11	·0121	·0016
4	74	·08	—·07	0049	80	·09	—·10	·0100	—·0014
5	19	·09	— 06	·0036	100	·10	—·09	·0081	·0030
6	5	·10	—·05	·0025	12	·12	—·07	·0049	·0020
7	90	·10	— 05	·0025	81	·12	—·07	·0049	—·0055
8	98	·10	— 05	·0025	16	·13	—· 6	·0036	·0060
9	21	·11	—·04	·0016	19	·14	—·05	·0025	—·0064
10	80	·12	—·03	·0009	5	·15	—·04	·0016	0030
11	2	·13	—·02	·0004	2	·16	—·03	·0009	·0006
12	43	·13	—·02	·0004	33	·17	—·02	·0004	·0024
13	77	·13	—·02	·0004	55	·17	—.02	·0004	·0004
14	16	·14	— 01	·0001	101	·17	—.02	·0004	·0006
15	54	·14	—·01	·0001	11	·18	—·01	·0001	·0014
16	12	·15	0	·0000	40	·18	— 01	·0001	·0000
17	33	·15	0	·0 00	77	·18	—·01	·0001	·0000
18	48	·16	+·01	·0001	36	·20	+·01	·0001	·0017
19	49	·16	·01	·0001	57	·21	·02	·0004	·0004
20	15	·17	·02	·0004	74	21	·02	·0004	—·0022
21	34	·17	·02	·0004	7	·22	·03	·0009	·0006
22	36	·18	·03	·0009	34	·22	·03	·0009	·0003
23	40	·18	·03	·0009	49	·23	·04	·0016	— 0003
24	30	·20	·05	·0025	54	·23	·04	·0016	·0055
25	11	·22	·07	·0049	79	·25	·06	·0036	—·0007
26	81	·23	·08	·0064	48	·26	·07	·0049	—·0056
27	93	·23	·08	·0064	30	·30	·11	0121	·0168
28	100	·24	·09	·0081	90	·30	·11	·0121	—·0081
29	79	·26	·11	·0121	21	·35	·16	·0256	·0066
30	7	·40	·25	·0625	93	·40	·21	·0441	·0075
	Sums	4·6	1·58	·1564	..	5·66	1·86	·1872	0268
	Averages	·154	·0526	..	..	·1886	·062	..	..
	S.D.			·073	..	..	..	·088	$r=·14$

(Explanation of the Table. *Order* denotes place of the students in the serial arrangement of the fatigue index values. *No.* is serial number of the students. *d* is deviation of a student from the average of the group. The figure at the bottom of the column under *d* gives the *a.d.* or average deviation of the group. *S.D.* or standard deviation of the group is the sq. root of the average of the squares of the separate deviations and has been obtained by the formula

$$=\sigma\sqrt{\frac{\Sigma(d^2)}{n-1}}\,.$$

xy is product of x, an individual's deviation from the average of the series with the right hand, and y, his deviation from the average of the left hand. r is coefficient of correlation between the two series of fatigue index, and has been obtained by the Pearson formula

$$r = \frac{\Sigma xy}{n \sigma_1 \sigma_2}$$

in which n is the number of students in both series, σ_1 the *S.D.* of the first series, and σ_2 the *S. D.* of the second series).

2. It would be interesting to compare fatigue index figures of students of particular ages with those of all the students taken together irrespective of their age differences.

Table 3.

A Comparative Statement of Fatigue Index Figures.

	AGE GROUP 17–23. 90 STUDENTS.		AGE GROUP 20. 30 STUDENTS.		AGE GROUP 19. 20 STUDENTS.	
	Right Hand.	Left Hand.	Right Hand	Left Hand.	Right Hand.	Left Hand.
Average.	·155	·188	·154	·189	·146	·163
A.D. ..	+ ·053	+ ·066	+ ·053	± ·062	± ·050	± ·065
S.D. ..	+ ·069	± 079	+ ·073	± ·088	± ·063	± ·083
P.E.$_{AV.}$	± ·0048	± ·0056	+ ·0087	± ·0100	± ·0094	± ·0121
C. ..	·446	·423	·474	·468	·431	·510
D. ..	·033		·035		·017	
P.E.$_{D.}$	± ·006		± ·013		± ·014	
r. ..	·36		·14		·13	
P.E.$_r$	± ·0628		± ·1202		± ·1473	

Explanation of the Table—*P.E.*$_{AV.}$ denotes probable error of the average, and has been obtained by the formula

$$P.E._{..} = \frac{0{\cdot}6745\ S.D.}{\sqrt{n}}$$

C. Stands for coefficient of variability of a series, and has been obtained by the formula,

$$C = \frac{S.D.}{Av.}$$

D denotes difference between the averages of two series $P{\cdot}E._D$ denotes probable error of the difference between the averages, and has been obtained by the formula

$$P.E._D = 0{\cdot}6745\sqrt{\frac{S.D_1^2 + S.D_2^2 - 2rS.D_1\,S.D_2}{n}}$$

It indicates the significant or chance nature of the difference between the averages. *P.E.* stands for probable error of the coefficient of correlation and has been obtained by the formula

$$P.E._r = 0{\cdot}6745\,\frac{1-r^2}{\sqrt{n}}$$

Judged by the various measures of variability, the first series of measurements is comparatively trustworthy. There is more uniform distribution in it than in the other two. The left hand series for the age group 19 is relatively more variable (cf. $P.E._{Av.} = \pm{\cdot}0121$ and $C = {\cdot}510$) than the other series. The figures for age group 20 approximate those of all the age groups, taken together, and this representative character of its figures agrees with the fact that 33 p.c. of the students examined belongs to this group alone. We cannot, however, draw the conclusion that the fatigue index of age 19 is really lower than that of age 20. That it appears so from table 2 may be due to the fact that the number of students examined under each of the groups 19 and 20 is very small. The low figures for the age group 19 are probably due to the absence of very high fatigue index values in it.

3. The fifty students under the age groups 19 and 20 were classified into three groups according to the strength of their maximal grips (1) strong (2) medium (3) weak. A man was considered to be of medium or normal strength of grip if he fell within the normal limits of the average of his group (the normal limits being here defined by the average deviation of the group). The average strength for the right hand of the 90 students is 40· k.g. and the *A.D.* is 4·8 k.g. Therefore those, whose strength of grip is within the limits of 40·4 ± 4·8 k.g. were classed under "medium" and those above the upper limit were classed under "strong" and those under the lower limit were classed under the "weak." In this classification the strength of the left hand grip was not taken into consideration. This has, no doubt, caused a slight error as the left handed persons should be judged by their left hands.

The fatigue figures for students of these three groups are given in Table 3A.

Table 3A.

Fatigue Index Figures for "strong", "average" and "weak."

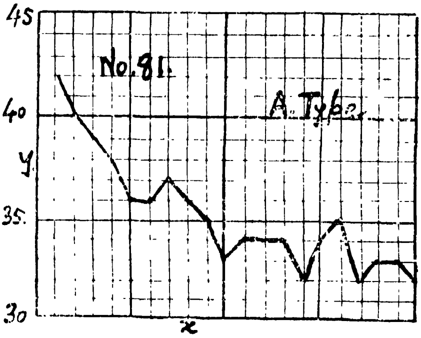
	Strong	Medium	Weak
R	134	150	177
L	192	184	157

(The figures indicate the average fatigue index.) The Table shows that the weak persons are more fatiguable in their right hands, but less so in their left hands and that as the fatigue index for the right hand decreases withthe increase of general strength, that for the left hand increases. The inverse correlation between fatigue index for the right hand and strength of grip indicated in Table 3 is also supported by the coefficient of correlation between these two traits in the 30 students of the age group 20 was found to be ·19 with a $P.E._r = \pm .1$. The r between the fatigue index and the strength of grip for the left hand of the same students was found to be only-·01. This appears to contradict the indication of positive correlation between the same two traits from the table 3 (i.e. fatigue index for the left hand and the strength of the left hand grip). But it is evident that all students and some more than others, are liable to the "fatigue" effect which the fatigue of the working right hand induces into the left. This induction of fatigue effect would to some extent account for the greater fatigue index figures for the left hand in Tables 1, 2 and 3.

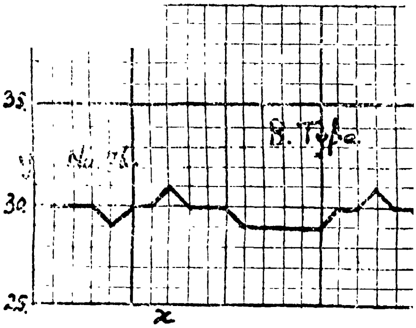
4. If a man continues gripping with his maximum effort for a certain length of time, he gradually reaches a level from which his power of endurance does not readily fall. There is in other words, a levelling tendency of fatigue. At the beginning of work individual differences are marked and the endurance curves present various sorts of irregularities. In the most common type of the endurance curves, and also to some extent in all endurance curves there is a tendency for the curves to rise and fall alternately. These alternate rise and fall seem to indicate the rhythmic character of "willed activities." The exertions come, as it were, in beats. These small changes re-

semble more or less the "short-time" oscillations of curves of economic and sociological phenomena.

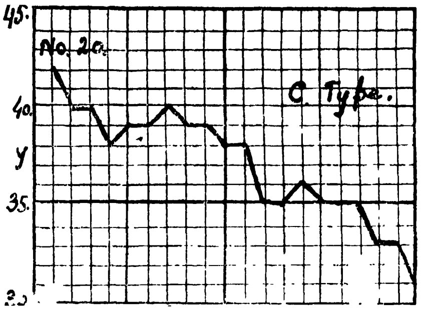
5. According to Binet and Vaschide 90 per cent of endurance curves can be put under four types: "(*a*) a sudden drop, then fairly constant; (*b*) an approximately stationary or constant type; (*c*) a continuous, but gradual drop; and (*d*) a more or less definite rise." They took five readings of grips with each hand alternately. We have generally taken 20 readings, so that our curves may be expected to be more valuable than theirs. So far as I have examined the curves of 60 students, I conclude that most of them can be classed under the above four types of Binet and Vaschide. I wish only to add a fifth type, viz. (*e*) steady and then sudden drop. This type of endurance is not easily detected from curves plotted from five readings. The five types are shown below:—



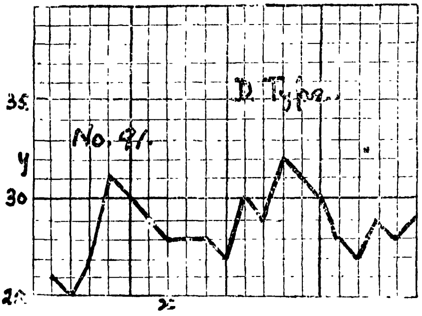
B—I.



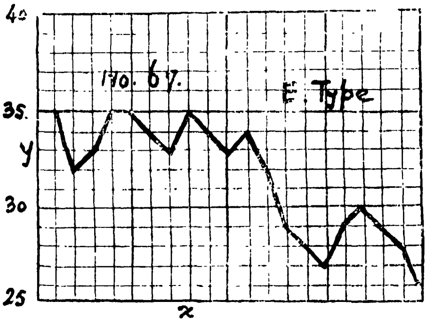
B—II.



B—III.



B—IV.



B—V.

6. Table 4 shows the manner of distribution of these five types of curves in the three groups of "strong" "medium" and "weak" subjects.

Table 4.

Distribution of Endurance Curves in the 3 *groups of Table* 3.

	Types	*A*	*B*	*C*	*D*	*E*
Strong (13 Students)	R	4		9		
	L	6	1	6		
Medium (30 Students)	R	6	7	14		3
	L	9	4	16		1
Weak (16 Students)	R	7	3	4	2	
	L	5	1	10		
TOTAL		37	16	59	2	4

According to Binet and Vaschide, *b* type seems to be the most common form but from our table it is only the subjects of average strength who are more of the *b* type than others. Vigorous individuals are more commonly of the *c* type. The specific characterestics of the several types cannot be considered now and shall be taken up afterwards.

7. The normal superiority of the right hand over the left has been indicated in our figures. The fact that *D* is more than five times $P.E._D$ in Table 2 (lower half, first column) shows that the difference of the averages of the fatigue index of the two hands is not due to chance, but is significant. The degree of scatter of the individual measurements from the averages as indicated by the respective '*A.D.*'*s* and *S.D.*'*s* is in all the three series greater for the left hand. The

less uniform character of the left hand series is indicated by their large *C.* and *P.E.AV.* in Table 2. The frequency curves of the fatigue indices for the right hand and for the left also indicate this. The curve for the right hand shows a slight skewness on the left side, whereas the curve for the left hand shows skewness on the right. The latter also is more irregular and multimodal than the former. Table 3 shows also that strong individuals have greater superiority of the right hand over the left, at least so far as fatigue index is concerned. The "weak" individuals have an inferior right hand in that sense. Table 4 also supports this conclusion. The "weak" individuals are more normal in their endurance curves with their left hands and less so with their right. Curves characterised by sudden initial drop are more common in their right hand. I conclude therefore, with Binet and Vaschide that physically feeble subjects have, as it were, "two left hands.' '

8. With a view to the analysis of the factors involved in fatigue, certain correlations were worked out. But many of them are not satisfactory as the number of measurements is very small and the probable influence of chance and systematic errors was not eliminated. These correlation figures require to be checked and corrected by further measurements in future. We may, however, note down the correlations (all worked out by the application of the "product moments method" of Karl Pearson). Without at present drawing any hasty conclusion from them: (*a*) I found an inverse correlation of —·19 between absolute strength of the right hand and the fatigue index of the same hand in age group 20. As all the students cannot be assumed to have gripped with their maximal effort each time a large amount of chance errors must have entered into our calculation. I thought that a more satisfactory coefficient of correlation would be secured by substituting the working maximum grip (i.e. the maximum grip recorded during 1 mt.'s work) for the absolute strength recorded separately and previously to the fatigue test. The two differ often by a large amount, as many subjects would not fully exert themselves during the work, though told to do so. The coefficient of correlation between this working maximal grip and fatigue index is —·25 for the right hand and —(·21) for the left (the latter figure much higher than the corresponding figure in the first calculation, viz. —·01).

(*b*) The work that a subject did by 16 grips was calculated and its relationship with his fatigue index was found by working out the coefficient of correlation between them. It is —·38 for the right hand of the age group 20. It would be interesting to compare this relationship with that found between work done and absolute strength as indicated by maximal grips of the right hand. The coefficient of correlation between work and strength of grip in the age group 20 is ·76. It is evident that the degree of a fatiguability of an individual lessens the total output of his work. But the positive influence which muscular strength has over the working capacity seems to be twice the negative influence which the degree of fatiguability has over it.

(*c*) I found a positive correlation of ·40 between weight and strength of grip for right hand in the age group 20 whereas I found no correlation between weight and fatigue index for the right hand, though a positive correlation of ·12 for the left hand. In extreme ranges of the series there was indication of greater positive correlation than in the middle ranges, where inverse correlations were rather conspicuous.

(*d*) The coefficients of correlation between fatigue index of the right hand and that of the left have been shown in table 2. Of the three *r*'*s*, *r* for the age group 17–23 is reliable, for it has a *p.e.* ,which is about $\frac{1}{6}$th times the *r*. As we have indicated above, subjects markedly differ among themselves as to the amount of fatigue effect induced from the right hand to the left in consequence of continuous exercise of the first while the second lies inactive. This unequal amount of induced fatigue effect introduces chance error in the measurements. The effect of this chance error has been probabiy minimised by counter balancing of opposite errors in the large series of 90 students of age groups 17–23 and hence the coefficient of correlation is greater among them. Besides this we may also note that as all the students share more or less in this induced fatigue effect it has acted as a constant error " constricting " the coefficient. We may, therefore, expect the *r* to be greater than even ·36.

§ 4.

Conclusions.

Before any definite conclusion can be drawn with regard to the general nature and conditions of fatigue, we must secure greater uniformity in the conditions of the investigation and base our results upon a larger number of measurements. What we can say at present with certainty is that the phenomena of physical fatigue and endurance involves a large number of complex factors and that the fatigue we are studying is fatigue of the organism only with reference to a certain set of special circumstances, and does not denote the general fatigue of the organism.

A GENERAL THEOREM IN THE GEOMETRY OF A PLANE CURVE.

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Introductory.

The following paper aims at an elementary exposition of the principles of certain new methods in the geometry of plane curves. It is written in the way of an introduction to the subject. It may prove interesting to students of Analysis Situs.

An important feature of the paper is the introduction of a system of new nomenclatures which may appear quite novel to many. The writer has found them convenient for his investigations. Some of them may commend themselves to mathematicians in general, such as "*the range of intimacy of one curve with another*," in place of "*the set of points of intersections of one curve with another.*"

It has been thought beyond the scope of this paper to enter into special applications. Some such applications of the methods will be found in the following papers by the author :—

1. "Geometrical Theory of a Plane Non-Cyclic Arc, Finite as well as Infinitesimal," *Journal of the Asiatic Society of Bengal,* (New Series), IV, No. 8, (1908).
2. "New Methods in the Geometry of a Plane Arc, I. Cyclic and Sextactic Points," *Bulletin of the Calcutta Mathematical Society*, I, No. 1, (1909).
3. "New Methods in the Geometry of a Plane Arc, II. Normals and Cyclic Points," *Bulletin of the Calcutta Mathematical Society,* X, No. 2, (1918–19).

1. Consider a *fixed* continuous plane arc S. Call it the *stem.* The two ends of the stem are A and B. Call A the *lower* end and B the *upper* end. The *positive* direction along the stem is from A to B.

At each point P of the stem suppose a tangent exists. The positive direction of this tangent at P is doing the positive direction of the stem at P. Suppose this positive direction of the tangent varies in a continuous manner as we proceed up the stem from A to B. The stem is free from cusps and nodes.

If the two points A and B coincide, the stem is *closed* and the point where the two ends meet is the *point of closure.*

An *oval* is a closed stem of which every point may be looked upon as the point of closure. The positive direction along the oval will be taken to be counter-clockwise.

2. Consider a *variable* curve T which crosses the stem S at a limited number of points $P_1, P_2, \ldots, P_s$. Call it the *tendril.*

We will suppose $P_1, P_2, \ldots, P_s$ are arranged in ascending order along the stem, so that P_1 is *above* A, P_2 is above $P_1, \ldots,$ and B is above P_s. We may also say A is *below* P_1, P_1 is below $P_2, \ldots, P_s$ is below B.

We will say that the tendril is *intimate* with the stem at $P_1, P_2, \ldots, P_s$, or that, $P_1, P_2, \ldots, P_s$ is the *range of intimacy* of the tendril with the stem. Two points P_m, P_{m+1} between which no other point of intimacy lies will be called *consecutive* points of intimacy.

In certain cases a selected number of consecutive points of intimacy $P_k, P_{k+1}, \ldots, P_{k+l}$ will be specially called *the points of intimacy* and the remaining points of intimacy which lie above or below these special points of intimacy will be called the points of *sub-intimacy.*

We will suppose the tendril to be a closed branch or a branch extending to infinity on both sides of some well-known algebraic curve of kind K of which the coefficients are freely or conditionally variable and which does not possess a node or a cusp. The order n of this curve as well as the assigned conditions to which the coefficients may be subjected will determine the kind K of the tendril.

The tendril of kind K will have index r if r arbitrary points of intimacy of the tendril with the stem suffice to determine the tendril uniquely.

The tendril may however be defined to pass through a certain number of *fixed* points in the plane besides the r variable points on stem. In general any r arbitrary points lying on the plane in addition to these fixed points, if they exist, will determine the tendril uniquely.

3. The following *conditions of intimacy* of the tendril with the stem will be supposed to hold. When these conditions hold the stem will be called *congenial* to the tendril

(i) The points of intimacy of the tendril with the stem have the same order and sense on the stem as on the tendril.

Suppose $P_1, P_2, \ldots, P_s$ are in ascending or positive order on the stem. Then $P_1, P_2, \ldots, P_s$ will also be in ascending or positive order on the tendril. We will say

The tendril embraces the stem in the same order and sense.

(ii) The tendril crosses the stem alternately from left to right and right to left.

As we proceed up the stem from A to B we will suppose that there is a continuous region to the right and a continuous region to the left of which the stem is the separating line. The tendril crosses the stem from left to right when it passes from the left region to the right region and it crosses from right to left when it passes from the right region to the left region. Between two consecutive crossings, the tendril, we will suppose, lies wholly in the same region.

A crossing of the stem by the tendril from left to right we will call a *positive* point of intimacy. And a crossing from right to left we will call a *negative* point of intimacy. Hence we may say

The range of intimacy of the tendril with the stem consists of elements of alternately contrary signs.

(iii) Two tendrils of kind K and index r cannot have more than $r-1$ points common in the stem or in a certain neighbourhood of the stem.

These $r-1$ points are exclusive of any fixed points through which the tendril may pass by definition. As r is the index of the tendril, two tendrils having r points common will be one and the same.

(iv) The tendril varies continuously with the r points of intimacy which suffice to determine it.

The tendril varies continuously in form and position as the r points of intimacy are varied in any continuous manner along the stem. In particular if the r determining points are taken in any interval δ of the stem which tends to vanish, the tendril will tend to a unique limiting form and position. The same may be said if the r determining points are divided into groups which lie in intervals tending simultaneously to vanish. The idea of continuity of variation involves the idea that the tendril does not split up or degenerate or develope nodes or cusps.

(v) The number of K points on the stem is limited.

A K point will be defined in the next article.

The stem will contain either no K points or a limited number of K points, separated by finite intervals. If there were an unlimited number of K points on the stem there would exist limiting points of of K points on the stem. The existence of these limiting points is impossible as the number of K points is limited

4. *A* range of $r+1$ points of intimacy of the tendril of kind K with the stem, taken in order with alternately contrary signs will be called *a K range*. The points of the K range are its *elements.*

If there be other points of intimacy lying between the extreme points of the K range besides those which belong to the K range they will be called *extra* points of the K range. These extra points will necessarily occur in pairs of elements of contrary signs lying between pairs of consecutive elements of the K range, for two consecutive elements of the K range are of contrary signs by definition and consecutive elements of the entire range of intimacy of the tendril with the stem are also of contrary signs. A K range which does not possess extra points will be called clear.

If there be other points of intimacy above or below the extreme points of the K range, they will be called *sub-extra* points.

The $r+1$ elements of the K range together with the extra points when they exist constitute the set of points of intimacy of the K range. The sub-extra points, when they exist, constitute the set of points of sub-intimacy of the K range. The set of points of intimacy of the K range together with the set of points of sub-intimacy constitute the entire range of intimacy of the tendril with the stem.

The interval of the stem, lying between two extreme elements of the K range is called the *interval* of the K range.

A part of the tendril lying between two consecutive points of the range of intimacy will be called a *loop* of intimacy. Loops of intimacy will be alternately on the right and left or left and right sides of the stem. A loop lying on the right will be called positive and a loop lying on the left will be negative.

A neighbourhood of a point O of the stem will be called ***upper,*** *lower* or *double* according as the neighbourhood extends to the upper, lower or both sides of O. The unqualified expression *neighbourhood of O* shall always mean a double neighbourhood of O.

A point O of the stem will be called a K point if every neighbourhood of O contains a K range of given sign. The K point would be positive or negative according as the corresponding K range is positive or negative. A positive K point will be written as $+K$ point and a negative K point will be written as $-K$ point.

A tendril is said to have contact of order p with the stem at O, if in every neighbourhood of O there are $p+1$ consecutive points of intimacy of the tendril with the stem. Thus at a K point, the tendril has contact of order r with the stem.

Imaginary points and so-called coincident points of intimacy do not count in our investigations. Whenever we say that a tendril has contact of order p with the stem at O we imply the actual existence of the set of $p+1$ real and distinct consecutive points of intimacy in every arbitrary neighbourhood of O. The contact position of the tendril is derived as a limit. It does not pre-exist in the logical order of thought. In the contact position, the tendril may be said to have *just left* intimacy with the stem rather than just gained, or we may say that in the contact position the tendril is just on the point of gaining intimacy with the stem. By adopting this point of view we shall avoid saying in any case that a number of points of intimacy of the tendril with the stem has coincided.

5. One K range is said to be higher than another K range if the elements of the former are higher than the corresponding elements of the latter with possibly some coinciding.

A continuous variation of the elements of a K range will be called a *proper variation* if—

(i) during the variation, the elements of the K range remain within the stem;

(ii) the elements of the K range as well as the extra elements of the K range when they exist or are developed maintain their relative order. Any consecutive two may come into as close a neighbourhood as one wishes but do not coincide with or cross each other. Extra elements when they exist or are developed do not disappear;

(iii) sub-extra elements of the K range when they exist or are developed may afterwards disappear, but do not coincide with or cross the extreme elements of the K range.

A proper variation of a K range will be called *elementary* if during the variation $r-1$ elements of the K range remains invariable and the other two elements vary.

An elementary variation will be called an elementary *contraction* if during the variation, the two variable elements continually approach each other.

A K range will be said to undergo a *progressive* contraction if it undergoes a series of elementary contractions in which each element moves in a constant direction or remains stationary during each contraction of the series.

If a set of consecutive elements of a K range are brought together by a proper variation within any arbitrarily small neighbourhood of O, they are said to *congregate* at O. A K point, for example, is a point at which all the $r+1$ elements of a K range congregate.

A set of consecutive elements are said to congregate *beside* O if they are brought into an arbitrarily small upper or lower neighbourhood of O. In the former case we will say they congregate *upside* O and in the latter case *downside* O.

A progressive contraction of a clear K range will be called *simple* if the elements of the K range divide into two groups, a lower and an upper which continually approach each other. The last two elements P_1 and P_{r+1} are the first to undergo an elementary contraction till P_1 (or P_{r+1}) congregates beside P_2 (or P_r). The congregation P_1P_2 and the element P_{r+1} are then made to approach each other by alternate simple contractions of P_2, P_{r+1} and P_1, P_{r+1} till the congregation P_1P_2 comes beside P_3 or P_{r+1} comes beside P_r. The process is continued in this manner. It will result in congregation of all the elements at a K point unless stopped at some stage. As soon as extra points are developed the process must stop or it may stop when all the elements on one side of an arbitrary fixed point O within the interval has congregated beside O

One K range is said to *cross* another K range, which is either higher or lower, if the interval of each contains in its interior an extreme element of the other.

Two cross ranges are said to have *external cross contact* if the elements of each range which lie in the common interval of the two cross ranges congregate beside each other, so that the common interval is arbitrarily small.

The cross ranges are said to have *internal cross contact*, if the elements of one range which lie in a non-overlapping part of its interval congregate beside the nearest extreme element of the other range, so that this non-overlapping part is arbitrarily small.

An interval of the stem will be called *free* if it does not contain any K point in its interior.

An interval of the stem will be called *prime* if it contains in its interior only one K point.

An interval of the stem will be called *composite* if it contains in its interior more than one K point.

A K range will be called *prime* if it does not possess any extra elements, neither does it develop any extra elements during any proper variation in its interval. A K range in a prime interval will be prime but the interval of a prime K range is not necessarily prime.

A K range which is not prime will be called *composite.*

Suppose a K range initially clear developes during a simple progressive contraction a pair of extra points. We can now reduce the range by considering the two highest or two lowest points of the range as sub-extra or by considering each of the extreme points of the range as sub-extra. In the fisrt case, the reduction is *unilateral* and in the second case the reduction is *bilateral.* A unilateral reduction is *infra-lateral* or *supra-lateral* according as the two lowest or the two highest elements of the range are reduced.

6. We will now establish some elementary theorems. The stem will be supposed to be congenial to the tendril.

Theorem I.—The sign of each element of a K range as well as of each extra element remains invariable during a proper variation.

If any element of the range of intimacy of the tendril with the stem change sign, then every element must change sign at the same time as consecutive elements of the range of intimacy are of contrary signs. This is impossible as the elements of a K range as well as the extra elements of the K range maintain their relative order during a proper variation and do not cross or coincide with each other. If all the elements of a range of intimacy change sign, then all the loops of intimacy change sign and in doing so must coincide with the stem at some stage. But a loop of intimacy cannot coincide with the stem as the number of points common to the tendril and the stem is always limited.

The only conceivable way in which an element P of a K range may change sign is when two extra elements are developed indefinitely close to P on either side. This case will be dealt with in the course of demonstration of the next theorem.

Theorem II.—Extra elements of a K range are developed in pairs between consecutive elements of the range.

Consider a K range initially clear of extra elements. The development of an extra element is preceded by the bending down of one of the loops of intimacy on the corresponding interval of the stem giving rise to a contact of the p^{th} order of the tendril with the stem at a point O which is either an interior point or an end point of the interval $P_k P_{k+1}$.

First suppose O is an interior point of $P_k P_{k+1}$. Then in an arbitrary neighbourhood of O falling within $P_k P_{k+1}$ there are developed $p+1$ extra points of intimacy. Now as the signs of P_k, P_{k+1} originally contrary continue to be so after the development of the extra points of intimacy by proper variation and as the extra points must obey the law of alternately contrary signs with the elements of the K range, they must be even in number.

Now suppose O is an end-point of $P_k P_{k+1}$. Say O is at P_k. Then in an arbitrarily small neighbourhood of P_k. there are developed $p+1$ points of intimacy of which one is P_k and the others are extra points. These $p+1$ points lie between P_{k-1} and P_{k+1} which are of the same sign. Consequently $p+1$ must be an odd number. Hence the number of extra points of intimacy developed will be even. This set of $p+1$ points of intimacy will be of alternately contrary signs. We can identify any of these of a sign contrary to that of P_{k-1} or P_{k+1} as the point P_k, so that between P_k and P_{k-1} as also between P_k and P_{k+1} there will be an even number of extra points of intimacy. If P_k be the lowest element of the K range then we can choose as P_k the lowest possessing suitable sign of the set of $p+1$ points; so that the new points of intimacy developed will consist of an even number of extra elements and a single or no sub-extra element. The same might be said if the point O were at P_k+1.

If the K range be not initially clear then the new extra points will be developed in pairs falling between pairs of consecutive elements of the K range for the old extra points by definition exist in pairs between consecutive points of the K range.

If extra elements are developed simultaneously at each of the $r+1$ points $P_1, P_2, \ldots, P_{r+1}$ of the K range and if the topmost and bottommost points developed have the same signs as P_{r+1} and P_1 respectively then we can identify them with P_{r+1} and P_1 and with suitable identifications at all the other points of the K range, the K range will maintain the signs of its elements inviolate and consequently the number of extra points developed between any two consecutive points of the K range will be even. If however the topmost or bottommost extra point differ in sign from P_{r+1} or P_1 then we can maintain the sign of P_{r+1} or P_1 inviolate by considering this extra point as sub-extra.

Theorem III.—In an elementary variation of a K range the two variable elements of the K range move in opposite directions and in general any two variable elements in the whole range of intimacy of the tendril which have between them no other variable element always move in opposite directions.

First, consider two variable consecutive elements P_k and P_{k+1} of the range of intimacy of the tendril with the stem. If possible suppose in an elementary variation P_k and P_{k+1} receive small displacements in the same direction, say upwards, to P'_k and P'_{k+1} where P'_k lies between P_k and P_{k+1}. Then the loops $P_k\ P_{k+1}$ and $P'_k\ P'_{k+1}$ are of the same sign and the intervals P_kP_{k+1} and $P'_k\ P'_{k+1}$ cross each other. Consequently the loops $P_k\ P_{k+1}$ and $P'_k\ P'_{k+1}$ must cross each other at some point. Thus two different tendrils of kind K having $r-1$ points common have another point common which is impossible.

Next, consider two variable elements P_k, P_l of the range of intimacy of the tendril with the stem which have between them only elements which are invariable. Suppose P_k and P_l are displaced to P'_k and P'_l by an elementary variation. The loops $P_k\ P_{k+1}$ and $P_{l-1}\ P_l$ where P_{k+1} and P_{l-1} are invariable elements must lie both within or both without the loops P'_kP_{k+1} and $P_{l-1}\ P'_l$ for every pair of corresponding loops of two tendrils having $r-1$ points common on the stem must possess this property. Hence if P'_k lie between P_k and P_{k+1}, then P'_l will lie between P_{l-1} and P_l and if P'_k be below P_k then P'_l will lie above P_l. Thus P_k and P_l will be displaced always in the same direction.

Lastly, suppose P_k and P_l are two variable elements of the K range which have between them no other element of the K range or invariable elements of the K range. If no extra elements of the K range lie between P_k and P_l, then the proof already given holds. If any extra elements exist between P_k and P_l, then they will exist in pairs. Suppose there is only one such pair $P_g\ P_{g+1}$. Then if P_k move downwards P_g will move upwards and consequently P_l will move upwards. Similarly if P_k move upwards P_l will move downwards. If there are more than one pair of extra points between P_k and P_l similar proof will hold.

Theorem IV.—In any proper variation of a prime K range it cannot happen that the elements of the K range are all displaced in the same direction or some are displaced in the same direction and the rest are invariable.

Suppose $P_1, P_2, \ldots, P_{r+1}$ are the initial positions of the elements of the K range. Suppose if possible all of them are displaced upwards by a proper variation to new positions P'_1 $P'_2, \ldots, P'^1_{r+1}$. Some however may be considered invariable. By a series of elementary variations of the range $P'_1, P'_2, \ldots, P'_{r+1}$ bring down P'_1 down to P_1 while all the other elements move upwards. Again apply a similar method to bring P'_2 down to P_2 while P'_1 remains invariable and all the other elements move upwards. By repetitions of the method all the elements except P'_r P'_{r+1} will have been brought back to their original positions and P'_r and P'_{r+1} will have both moved further upwards from P_r and P_{r+1} which is impossible by Theorem III.

Theorem V.—A prime K range converges to a unique K point.

By a simple progressive contraction the interval of a K range can evidently be made to acquire a sequence of diminishing values converging to zero, each interval lying within the preceding one. The sequence of intervals define a certain point O on the stem which is common to all the intervals. In every neighbourhood of this point O there is a K range. Therefore the point O is a K point of the same sign as the given K range for a K range maintains its sign during a proper variation.

This K point O is unique. If possible suppose by some other method the K range converges to some other point O' on the stem where O' is above O. Take two sufficiently small neighbourhoods about O and O' which do not overlap. Then there is a K range in each of these neighbourhoods such that one is a proper variation of the other. This is impossible by Theorem IV, as in that case all the elements of the K range about O will have moved upwards to the neighbourhood of O' by a proper variation.

Theorem VI.—A K point cannot at the same time be both positive and negative.

In a positive K range the tendril crosses from left to right at the lowest point of the range. Hence in the limiting form to which the tendril tends as the elements of the K range converge to the corresponding K point, the tendril approaches the stem from the left side. Similarly at a negative K point, the tendril approaches the stem from the right side. Now as the limiting form to which the tendril tends as the $r-1$ determining points of intimacy approach each other is unique, we see that the given K point cannot at the same time be positive as well as negative.

But it may be argued that at a particular point O, the tendril may have a contact with the stem of order $r+1$. In this case the tendril should have in every arbitrary neighbourhood of O, $r+2$ points of intimacy with the stem. Of these $r+2$ points of intimacy if we take the first $r+1$ we shall have a K range of a given sign, say positive. If we take the last $r+1$ points we shall have a K range which is negative. Consequently it may be argued that at the point O, there exists both a positive and a negative K point. But a little consideration will show that such a contingency is impossible. From a purely geometrical point of view a contact of the $r+1^{th}$ order at O implies the existence of $r+2$ real points of intimacy in every arbitrary neighbourhood of O. Now if we try by a simple progressive contraction to make the first $r+1$ points to converge at O, the $r+2^{th}$ point will be continually moving away from O, so that if the interval in which the $r+2$ points existed at any moment was progressively contracted it would soon cease to hold the $r+2^{th}$ point.

Again suppose we have an unlimited number of K points in the stem. These will be alternately positive and negative as we shall prove later on. Suppose O is a limiting point of these K points. Then in every neighbourhood of O, there will be a positive K point as well as a negative K point and consequently a positive K range as well as a negative K range. In this case the point O might be called a positive as well as a negative K point. This contingency does not however arise as we have supposed the number of K points on a stem to be always limited. (Vide, condition V of congeniality.)

This theorem is fundamental to our investigations.

Theorem VII.—If a composite K range undergo a progressive contraction with unilateral reductions it will ultimately converge to a K point of the same sign as the original K range.

Suppose we start with a K range initially clear of extra points and apply to it a simple progressive contraction with unilateral reductions whenever a pair of extra points are developed. This unilateral reduction will not alter the sign of the K range. Repeat this process continually. Then a certain stage will be reached after which simple progressive contraction will no further develop extra points.

For if the development of extra points continued indefinitely while the interval of the K range converged to a point O, then in every neighbourhood of O there would be a K range with extra points. This K range with extra points by unilateral and bilateral reductions would give rise to two K ranges with different signs. Consequently the point O would be both a positive and a negative K point which is impossible.

Thus every K point converges by simple progressive contractions with unilateral reductions to at least one K point of the same sign which is interior to is interval. The unilateral reductions we will suppose always supra or always infra although the argument does not require it.

Theorem VIII.—Every K point has a neighbourhood in which the corresponding K range is prime.

Take any prime neighbourhood of K; there must exist a K range of the same sign as that of K in this neighbourhood. This K range will be prime, for if by any proper variation in the prime interval, a pair of extra points are developed, then by bilateral reduction we shall get a K range of the opposite sign which will converge to a corresponding K point. This latter K point being of a sign different from that of the given K point must be a point different from it. Consequently there are two K points in the same prime neighbourhood which is impossible.

Theorem IX.—The K points of a stem are alternately positive and negative.

Suppose O and O' are two consecutive K points on a stem S, O' being above O. Suppose O is a $+K$ point. Take any prime neighbourhood of O, this neighbourhood will not contain O' as an interior point O. Any K range $P_1, P_2, \ldots, P_{r-1}$ in the neighbourhood of O' will be positive. That is the point P_1 will be positive. Some of the elements of this range will be above O, others below O. We can transfer the element nearest to O on the downside by a simple progressive contraction of the K range in which the remaining elements on the downside of O remains invariable. By repeating this process we can transfer all the elements on the downside of O except the last element to the upside of O.

Take any prime neighbourhood of O' with corresponding K range $P'_1, P'_2, \ldots P'_{r+1}$. We can transfer all the elements $P'_1, P'_2, \ldots, P'_r$ to the downside of O while P'_{r+1} remains on the upside of O. Now the interval OO' is free. We can therefore transfer $P_2, P_3, \ldots, P_{r+1}$ to $P'_1, P'_2, \ldots, P'_r$ respectively without development of any further points of intimacy, for in a prime interval there cannot exist more than r points of intimacy. Consequently $P_2, P_3, \ldots P_{r+1}$ will carry their signs with them when they are transferred to $P'_1, P'_2, \ldots, P'_r$. But the tendril is determined by the r points of intimacy. Therefore the signs of P'_1 and P'_1 are contrary. And hence the K points O and O' are of contrary signs.

Cor.—In an oval there are always an even number of K points for they are of alternately contrary signs.

Theorem X.—If of two prime K ranges of opposite signs one be above the other, then the point of convergence of the first is above the point of convergence of the second.

The two K ranges being prime and of opposite signs will converge to two distinct and unique K points of opposite signs. If the two K ranges be separate, that is if every element of the first be above every element of the second, with possibly the lowest element of the first coinciding with the highest element of the second, then evidently the K point to which the first converges is above the K point to which the second converges, as the K point corresponding to each K range is on interior point of its interval.

It is only in the case where the two K ranges cross each other that the theorem requires proof.

Suppose the first range is $P_1, P_2, \ldots, P_{r+1}$ which is above the second range $Q_1, Q_2, \ldots, Q_{r+1}$. Apply a simple progressive contraction to the range $P_1, P_2, \ldots, P_{r+1}$ till the elements of the range below Q_{r+1} congregate on the downside of Q_{r+1} or the elements above Q_{r+1} congregate on the upside of Q_{r+1}. It may be observed that during this simple progressive contraction of the first range, the first range continues to be above the second range.

In the first case the two ranges will have external cross contact and a progressive contraction applied to the second range will separate the two ranges and the theorem will follow.

In the second case the two ranges will have internal cross contact. Now apply a simple progressive contraction to the second range, till the elements of the second range above P_1 congregate on the upside of P_1 or the elements of the second range below P_1 congregate on the downside of P_1.

In the first case the two ranges will have external cross contract and can be separated by a further simple progressive contraction given to the first range.

In the second case the two ranges will have internal cross contact. By continual application of simple progressive contractions alternately on the two ranges they will either separate or continually contract and converge to a common point O, which will be thus both a positive and a negative K point, which is impossible.

Cor.—If $P_1, P_2, \ldots, P_{r+p}$ be $r+p$ consecutive points of intimacy of the tendril with the stem and if the ranges $P_1 \ldots, P_{r+1}, P_2 \ldots P_{r+2}, \ldots, P_p \ldots P_{r+p}$ be all prime, they will converge to P unique K points of alternately contrary signs.

Theorem XI.—A composite K range converges to a highest and a lowest K point which have the same sign as the original K range.

Suppose we start with a K range initially clear and apply to it progressive simple contraction. At some stage it will develop a pair of extra points. By infra and supra reductions we shall respectively get two K ranges of the same sign which cross each other, the first being above the second. If we go on applying progressive simple contractions with infra reductions to the first range we shall get the highest K point of the range and if we go on applying progressive simple contractions to the second range we shall get the lowest K point of the range.

If we adopt the method of cross contact explained in Theorem X to the two cross ranges with infra reductions to the first and supra reductions to the second they would always continue to be cross, that is, the first will continue to be higher than the second with a common interval between them or they will separate.

If they do not separate at all then they will ultimately converge to a common K point in every neighbourhood of which there will be two K ranges of the same sign which cross each other of which one is necessarily higher than the other. This impossible (Theorem IV) as ultimately the neighbourhood will be prime.

Cor. 1.—Every composite K range converges to at least three K points, as between the two extreme K points of the same sign there is a K point of the opposite sign. Theorem IX.

Cor. 2.—If two composite K ranges of contrary signs cross each other, they will converge to at least four K points.

We will now enunciate a general theorem of importance.

Theorem XII.—If an oval tendril of kind K and index r, have $2p$ $(\geqq r+1)$ points of intimacy with an oval stem congenial to the tendril, then there will exist on the oval at least $2p$ distinct K points on the stem.

Suppose $P_1, P_2, \ldots P_{2p}$ are the $2p$ points of intimacy. They form $2p$ successive K ranges $P_1 P_2 \ldots P_{2r+1}$, $P_2 P_3 \ldots P_{r+2}, \ldots$ $P_{2p} P_1 \ldots P_r$ of which any two consecutive ones are of opposite sign and cross each other.

If all the ranges be prime, then by Theorem X, they converge to $2p$ unique K points of alternately contrary signs and the stem will contain exactly $2p$ distinct K points.

If some or all ranges be composite, the number of K points to which they will converge will be generally greater. A complete discussion of this case is beyond the scope of this introductory paper.

DIRECT REPLACEMENT OF NEGATIVE GROUPS BY HALOGEN.

SURENDRA CHANDRA DHAR, M.SC.

There are numerous instances of replacement, elimination and migration of different groups and atoms in aromatic series. Systematic study has been made in the Diazo compounds by a number of workers, of whom the most prominent are Meldola, Hantzsch, Orton and Morgan * and definite laws have been established with respect to such intramolecular changes. Work, though of much less comprehensive character, has also been done in connexion with migration and elimination of groups and atoms in acetylated and halogenated phenols †

These are only generalized instances of the mobility of groups and atoms in aromatic series, but there is hardly any substituent group or element which is too strong to be replaced by an entrant group or element, or too weak to enter the ring by driving out other groups or elements.

There are isolated instances of the direct replacement of negative groups by halogen; and as it has been found to be of great practical and theoretical interest as well, it is proposed to study this reaction systematically.

At present nitro compounds which form a very important class have been taken up and will be treated in some detail, but a few instances from previous work on the direct replacement of two other important negative groups such as carboxyl and sulphonic acid, may not be out of place.

Carboxyl (-COOH) *group replaced by halogen.*

1. Benzoic acid when heated with bromine and iron in a sealed tube gives hexabromobenzene.‡

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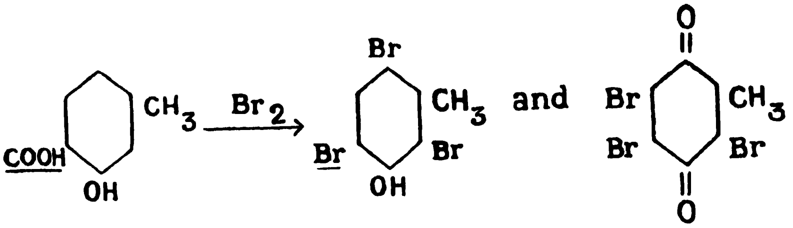
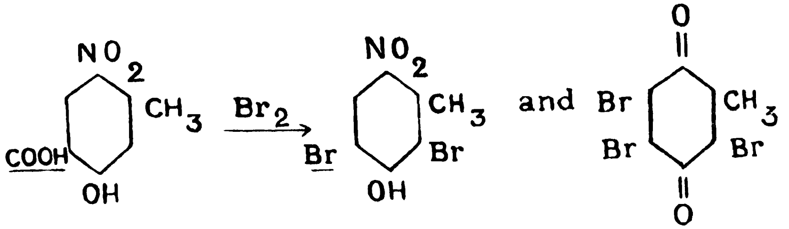
* *The Chemistry of the Diazo Compounds* (J. C. Cain, pp. 63–69).

† *Ann. 364*, 147-182 [1909]; *Ber. 47*, 1297-1304, [*1914*] and *T. 1908*, 793; *1912*, 1964; *1914*, 111 and 1885.

‡ *J. Amer. C. S. 19*, 365; *T. 1914*, 1887.

3. In an article on "Bromination and nitration in the aromatic series" Blanksma has arrived at a generalization "that a carboxyl group in an *o*- or *p*- position" to NHCH$_3$ group in the benzene series "is replaced by Br or NO$_2$ with evolution of CO$_2$.*





Sulphonic acid (—SO$_3$H) *group replaced by halogen.*

1. In anthraquinone series many halogenated compounds have been prepared by direct replacement of sulphonic acid group by halogen, e g.

(*a*) 1 : 8 and 2 : 7 Dichloro-anthraquinones from corresponding disulphonic compounds.†

(*b*) 1 : 5 Dibromo-anthraquinone from 1 : 5 disulphonic compound.‡

2. Sulphonic acid group in aromatic compounds in general has been replaced by chlorine under the influence of thionylchloride.§

e.g.

$$C_6H_5SO_3H \xrightarrow{Cl_2} C_6H_5Cl$$

$$Cl\,C_6H_4SO_3H \xrightarrow{Cl_2} ClC_6H_4Cl.$$

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* *Proc. K. Akad. Wetensch.* Amsterdam *1902*, 643–647.

† *D.R.P* 205913–228876.

‡ *D.R.P.* 205145.

§ *D.R.P.* 280739, *Monatsh.*, *36*, 719–722 ; *86*, 723–730.

Replacement of nitro ($-NO_2$) group by halogen.

1. In 1866, Kekule heated nitrobenzene in sealed tubes with bromine and got tribromo-, tetrabromo- and pentabromobenzene.[*]

2. Nitrobenzene under the influence of thionylchloride gives chlorobenzene.[†]

3. Meta-Dinitrobenzene with chlorine gives m. chloronitrobenzene and finally m. dichloro benzene.[‡]

4. Prof. Armstrong says that picric acid with bromine gives the same dinitrobromophenol as dinitrophenol gives with bromine [§]

5. Dinitro-o- and dinitro-p-anisidine on diazotisation in hydrochloric acid give substituted diazonium compounds where one of the two nitro groups has been replaced by chlorine.[‖]

6. α or β Nitro-anthraquinone with chlorine gives α or β chloro-anthraquinone.[¶]

1 : 5, 1 : 6 and 1 : 7 Dinitroanthraquinones similarly give the corresponding dichloro-anthraquinones.[**]

7. A few instances of the replacement of nitro group by bromine have been found in nitrated diphenylamines.[††]

Only a few typical nitro compounds from benzene, naphthalene, anthracene, xanthone and coumarin series have been tried and of halogens, only bromine has been used ; other nitro compound as well as aromatic compounds with other negative groups are under investigation.

Benzene Series.

1. Picric acid with three molecular proportions of bromine in a sealed tube at 130°, 250° or 300°C for six hours gives tetrabromophenol, pentabromophenol and hexabromobenzene. Hexabromobenzene is proportionately less at lower and greater at higher temperature.

Pentabromobenzene bromide.

2. Trinitrotolene (T.N.T.) with three molecular proportions of bromine at 200° for eight hours gives pentabromobenzylbromide. Fine white needles from benzene melting at 320°. When heated with abso-

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[*] *Ann. 137*, 169.

[†] *Manatch 86*, 723–730

[‡] *D.R.P.* 108165.

[§] *J. Soc. Dyers Colourists*, 1885, June. p. 185.

[‖] French patent No 315932

[¶] *D.R.P.* 252578.

[**] *D.R.P.* 280739 and *Monatsh.*, *36*, 719–722 and *86*, 723–730.

[††] *Proc. Roy. Irish. Acad.*, 1919, *34*, 218–225.

lute alcohol and potassium acetate it gives pentabromobenzyl alcohol. White needles from alcohol melting at 175°. Attempts have been made without success to oxidise pentabromobenzylbromide to pentabromo benzoic acid. Perhaps the same protective influence which prevents 2 : 3 i 6 tribomotoluene from being oxidized to the corresponding tribromobenzoic acid* is working here.

Napthalene Series.

1 : (4) : 5 : 8-Tetrabromonaphthalene.

3. 1 : 5 Dinitronaphthalene with two molecular proportions of bromine at 200° for six hours gives one tetrabromonaphthalene. White needles from benzene melting at 308°.

4. 1 : 8 Dinitronaphthalene under exactly the same conditions gives the same tetrabromo compound. The positions of three bromine atoms are 1, 5 and 8 by replacement of nitro groups in those positions and perhaps the fourth bromine atom is in position 4.

Xanthone Series.

5. α or β-Dinitroxanthone with two molecular proportions of bromine at 300° for ten hours gives tetrabromoxanthone †

6. *Dinitrodibromoxanthone.*—α_1 Tetra-nitroxanthone obtained by nitrating α—dinitroxanthone with four molecular proportions of bromine at 300° for twenty hours gives dibromo linitroxanthone. White needles from benzene melting at 265°.

β_1 Tetranitroxanthone (from β dinitroxanthone) gives the same dinitrodibromo compound under same conditions.

7. *Tetrabromoxanthone.*—α_2 Tetranitroxanthone (from α—dinitroxanthone) with four molecular proportions of bromine at 300° for twenty hours gives tetrabromoxanthone Yellowish white stuff from benzene melting at 255°.

8. *Hexabromoxanthone.*—β_2 Tetranitroxanthone (from β dinitroxanthone) with four molecular proportions of bromine at 300° for twenty hours gives hexabromoxanthone. White needles from benzene melting at 285°.

9. *Octabromoxanthone.*—Tetrabromodinitroxanthone when heated with excess of bromine in a sealed tube at 300° for twenty-six hours

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‡ *T. 1914*, 514.

† *T. 1916*, 745.

gives octabromoxanthone. White needles were obtained from benzene which do not melt up to 320°.

Anthraquinone Series.

10. 1 : 3 Dinitroanthraquinone with two molecular proportions of bromine at 300° for six hours gives 1 : 3 dibromoanthraquinone.

11. 1 : 5 Dinitroanthraquinone with two molecular proportions of bromine under same conditions gives 1 : 5 dibromoanthraquinone.

Coumarin Series.

12. 6-Nitro coumarin with bromine at 130° for six hours gives 3 bromo* 6 nitro coumarin.

13. 3 : 6 Dinitro coumarin with bromine at 130° for six hours gives 3 bromo—6 nitro coumarin.

14. *.6 Nitro—3 : 2 : 2 tribromo coumarin.*—3 : 6 Dinitro coumarin with bromine at 200° for six hours gives mononitro—tribromo coumarin. Greyish white needles from alcohol melting at 180°. As this compound is obtained at a higher temperature from 3 : 6 dinitro coumarin which at a lower temperature gives 3 bromo—6—nitro coumarin, most probably the nitro group is in position 6 and one of the three bromine atoms is in position 3. This supposition is further strengthened by the fact that this compound is different from mononitro 3 : 6 : 8 tribromo coumarin.†

15. *6 Nitro 3 : 2 : 2 : 2 tetrabromo coumarin.*—3 : 6 : 8 Trinitro coumarin when heated with bromine at 200° for six hours gives one mononitrotetrabromocoumarin. Greyish white substance from benzene melting at 260°. This is different from mononitro 3 : 6 : 8 : 2 tetrabromo coumarin.

A trace of iodine has all throughout been used as a catalyser and the experiments have been conducted in sealed tubes. Three molecular proportions of bromine have been taken throughout.

The results are tabulated below.

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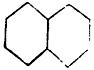
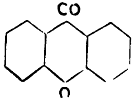
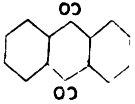
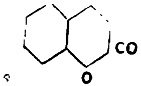
* This is the mean percentage.

† *Die coumarine*, H. Simonis. p. 171.

Series.	No.	Substance.	Quantity of Bromine.	Temperature.	Products.	Approximate yield (of theory).
Benzene	1	$C_6H_2(NO_2)_3OH$	3 Mols	250°	$C_6HBr_4(OH)$, C_6Br_5 (OH) and C_6Br_6..	70%*
	2	$C_6H_2(NO_2)_3CH_3$	3 ,,	200°	$C_6Br_5CH_2Br$	50%
 **Napthalene**	3	$1:5-C_{10}H_6(NO_2)_2$	2 ,,	200°	$1:(4):5:8-C_{10}H_4Br_4$	60%
	4	$1:8-C_{10}H_6(NO_2)_2$	2 ,,	200°	$1:(4):5:8-C_{10}H_4Br_4$	60%
 **Xanthone**	5	α or $\beta-C_{13}H_6O_2(NO_2)_2$ ..	2 ,,	300°	$C_{13}H_4O_2Br_4$	90%
	6	α_1 or $\beta_1-C_{13}H_4O_2(NO_2)_4$ ..	4 ,,	300°	$C_{13}H_4O_2(NO)_2Br_2$	80%
	7	$\alpha_2 C_{13}H_4O_2(NO_2)_4$	4 ,,	300°	$C_{13}H_4O_2Br_4$	90%
	8	$\beta_2 C_{13}H_4O_2(NO_2)_4$	4 ,,	300°	$C_{13}H_2O_2Br_6$	90%
	9	$C_{13}H_2O_2(NO_2)_2Br_4$ ..	excess	300°	$C_{13}O_2Br_8$	80%
 **Anthraquinone**	10	$1:3-C_{14}H_6O_2(NO_2)_2$ ..	2 ,,	300°	$1:3-C_{14}H_6O_2Br_2$	90%
	11	$1:5-C_{14}H_6O_2(NO_2)_2$ ..	2 ,,	300°	$1:5-C_{14}H_6O_2Br_2$ ..	90%
 **Coumarin**	12	$6.\ C_9H_5O_2NO_2$	3 ,,	130°	$C_9H_4O_2(NO_2)Br-6:3$	70%
	13	$3:6-C_9H_4O_2(NO_2)_2$ ..	3 ,,	130°	$C_9H_4O_2(NO_2)Br$	70%
	14	$3:6-C_9H_2O_2(NO_2($ ..	3 ,,	200°	$C_9H_2O_2(NO_2)Br\text{-}Br_2$	50%
	15	$3:6:8-C_9H_3O_2(NO_2)_3$ ..	3 ,,	200°	$C_9HO_2(NO_2)Br\text{-}Br_3$	30%

The first and the most important point to note is the economy of bromine whenever both replacement and bromination have taken place. In none of the instances cited from previous work excepting the first one, both replacement and bromination have taken place. So we shall discuss the first instance only, because the rest has no bearing on the present topic.

Kekule took 17 grams of nitrobenzene and 55 grams of bromine and got tribromobenzene, tetrabromobenzene and pentabromobenzene (loc. cit). Some nitrobenzene is left unacted; and nitrogen and a very small amount of hydrogen bromide were found along with the products of reaction. The presence of nitrogen led him to explain the reaction thus :—

$$2\ C_6H_5No_2+5\ Br_2=2\ C_6HBr_5+4\ H_2\ O+N_2.$$

But neither the whole of nitrobenzene was found to be converted into pentabromobenzene as it is represented by the equation nor were the respective yields of the three bromo compounds stated which might have to a certain extent given an idea with respect to the course of reaction.

Experiments are being conducted to follow the course of reaction by studying it quantitatively, by the isolation of intermediate products and by the employment of probable intermediate brominating agents. So it is quite premature to attempt to explain it on one supposition or another. But one thing is quite clear. From the first five instances (chart) where both replacement and bromination have been complete, it can safely be assumed that at least more than half the number of bromine atoms used, have been utilised to form the products of reaction.

In the case of picric acid (1) only six bromine atoms have been taken and one of the products of reaction is hexabromo benzene where all the six bromine atoms have entered into the ring. The yield is about 70 per cent of theory. In the usual process of bromination we require four bromine atoms to replace the two free hydrogen atoms in picric acid leaving only two bromine atoms for replacement of three nitro groups and one hydroxyl group. So it is clear that the reaction is quite different from the ordinary process of bromination. In the case of trinitro toluene (2) also, six bromine atoms have been taken and the product is pentabromo benzyl bromide, a hexabromo compound. In the case of dinitro-naphthalenes (3 and 4) four bromine atoms have been taken and the product is a tetrabromo compound and the yield is about sixty per cent of theory. In the case of dinitroxanthones (5) four atoms of bromine have been taken and the product is a tetrabromo compound and the yield is almost quantitative.

The second important point in is that none of the substances taken from benzene and xanthone series can be brominated under ordinary conditions whereas it has been possible to get bromine atoms in all the possible positions in the rings and in the case of trinitrotoluene one atom has gone also to the methyl group which has been found to be most resistant to halogen.*

Also the dinitroanthraquinones (10 and 11) cannot be brominated under ordinary conditions. But here the bromo compounds have been obtained almost quantitatively by replacement. No bromo derivatives of the naphthalene (3 and 4) and coumarin compounds (14 and 15) are known but each of them has given a polybromo compound by this reaction quite easily.

Lastly comes the question of orientation. This reaction is expected to help orientation immensely even after making reasonable allowance for migration of groups and atoms.

In the anthraquinone series (10 and 11) the bromine atoms have gone to the positions previously occupied by nitro groups. Example 13 is another clear proof in this line. In tetrabromophenol (1) one of the two free hydrogen atoms of the trinitro phenol is left, so bromine replaced the nitro groups from their real positions and not from the possibly migrated ones. In naphthalene series the tetrabromo compound obtained is not known, so the positions 1, 5 and 8 are probable and 4 is only possible. In xanthone series the positions of the nitro groups are not definitely settled so the question of orientation does not arise there.

In the coumarin series it is remarkable to find that nitro group in position 6 is not replaced and bromine enters the position 3 first whether it is free (or 12) or occupied by a nitro group (13, 14 and 15). This is however not unexpected as in nitration and bromination of coumarin. The positions attacked are 6 and 3 respectively† in preference to other positions.

Thanks of the writer are due to Prof. J. F. Thorpe and Dr. M. A.

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* *T. 1914*, 514.

† *T. 1910*, 1388–1408.

Whiteley, for the interest they are taking in this investigation, and to the Chemical Society for a grant from their Research Fund which has met part of the expenses incurred in carrying out this piece of work.

ESQUISSE D'UNE THÉORIE DES ENSEMBLES ABSTRAITS.

Par MAURICE FRÉCHET,
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INTRODUCTION.

Le présent travail est un résumé systématique des travaux que j'ai publiés ou qui sont parvenus a'ma connaissance depuis ma Thèse Afin d'abréger, je renvoie pour les démonstrations aux articles originaux.

Dans la *première partie* de ce mémoire, nous avons cherché surtout à indiquer la *genèse des idées* qui ont conduit aux définitions et aux généralisations auxquelles est consacrée la seconde partie. Dans cette *seconde partie, exposée de façon synthétique* et classée dans l'ordre des complications croissantes, on trouvera donc beaucoup de redites. Mais de cette façon, elle se suffit à elle même et un lecteur qui ne désire que faire rapidement connaissance avec les résultats acquis pourra se dispenser de lire la première. Si au contraire la seconde partie lui parait disposée de façon un peu artificielle, si un aide-mémoire ne lui suffit pas, il pourra se rendre compte dans la première que ces développements ne sont pas le fruit de l'arbitraire, qu' ils résultent du désir collectif d'unifier et de coordonner un certain nombre de généralisations faites dans des directions diverses, mais *imposées* par les progrès de l' Analyse.

Remarque.—Ceux des lecteurs qui ont eu l'occasion de parcourir ma Thèse ou quelques autres de mes publications devront prendre garde qu' à la suite des progrès effectués, j'ai été amené à *modifier* quelques unes des *notations* et des *définitions* employées dans ma Thèse.

PREMIÈRE PARTIE.

L' EVOLUTION DE LA NOTION DE LIMITE.

1. *Point-limite.*—La notion de limite s' est introduite de bonne heure dans les sciences mathématiques. C' est ainsi que sa considération s' impose dans la sommation des progressions géométriques convergentes liée au fameux paradoxe de Zénon.

Si l'on utilise l'image géométrique qui fait correspondre à un nombre un point d'une droite ayant pour abscisse ce nombre, on pourra préciser ainsi cette notion. Une suite infinie S de points d'une droite $P_1, P_2, \ldots P_n \ldots$ est une *suite convergente* s' il existe un point P de la droite tel que la distance PP_n devienne et reste à partir d'un rang n assez élevé aussi petite que l'on veut. Le point P est alors dit le *point-limite* de la *suite convergente.*

2. Pendant des siècles une telle définition de la limite a suffit à tous les besoins des mathématiques. Elle s' est étendue sans peine à des classes d'éléments plus générales. C'est ainsi qu'on peut conserver exactement la même définition dans les mêmes termes si l'on substitue aux mots points de la droite, les mots points du plan points de l'espace, etc...

3. *Point d'accumulation.*—Le développement de l'analyse dans le siècle précédent a cependant nécessité l'élargissement de la notion de point-limite d'une suite convergente.

On peut dire que l'une des raisons principales qui nécessitent l'introduction des limites est la suivante : une certaine propriété ayant lieu pour une suite convergente de points P_1, P_2,...., il arrive fréquemment que le point limite P posséde la même propriété. D'autre part il est souvent plus facile de prouver que P est point limite de la suite que de prouver qu' il a la propriété considérée.

Or il arrivera souvent que l'ensemble E de points satisfaisant â une propriété déterminée ne forme pas une suite convergente. Si l'on veut cependant appliquer le procédé que nous venons d'indiquer, on sera amené à considérer comme jouant vis â vis de l'ensemble E de points le même rôle que le point P jouait relativement à la suite S de points P_1, P_2,...., tout point Q qui est point limite d'une suite convergente de points Q_1, Q_2, Q_3....distincts et extraits de E. Il était assez naturel de dire que Q serait point-limite de l'ensemble E et c'est ce qui a été fait par la plupart des auteurs. Certains néanmoins ont senti le besoin de marquer qu'il y avait là une idée nouvelle et distincte, et ont employé le mot de *point d'accumulation* d'un ensemble. C'est par exemple ce que fait Jules Tannery dans son *Introduction à la Théorie des fonctions d'une variable*; et bien qu'ayant jusqu'ici employé moi-même l'expression point limite, je me rallie à l'emploi de ce nouveau terme.

C'est l'étude des discontinuités des fonctions d'une variable réelle, instituée à propos de la théorie des séries de Fourier qui semble avoir été la raison d' être de l'introduction des points d'accumulation des ensembles linéaires (c'est à dire des ensembles de points d'une droite). Dans le plan, l' étude des singularités des fonctions analytiques a montré combien la notion de point d'accumulation d'un ensemble plan était indispensable : on sait en effet que tout point d'accumulation d'un ensemble connu de points singuliers est aussi un point singulier.

4. *Distance.*—Mais le développement de l'analyse conduisait aussi à une extension de la notion de limite dans une autre direction. La conception de classes d'éléments autres que les points de l'espace à 1, 2 ou 3 dimensions suscitait d'abord une *extension de la notion de distance.* C' est ainsi qu' on peut définir la distance de deux fonctions $f(x)$, $\phi(x)$ dont les carrés sont intégrables sur un même intervalle x_0, x_1 comme la valeur de l'expression

$$\sqrt{\frac{1}{x_1 - x_0}\int_{x_0} [f(x) - \phi(x)]^2 dx}$$

Cette définition ne se présente pas comme une généralisation arbitraire ; c' est elle qui fait comprendre l'importance de la série de Fourier d'une fonction $f(x)$ de carré intégrable. La somme d'un nombre déterminé de termes de cette série se présente en effet parmi les sommes analogues avec des coefficients différents comme étant celle dont la distance à $f(x)$ est minimum. D'autre part, elle jouit des proprtétés essentielles de la distance euclidienne ; c' est à dire que si l'on dèsigne par (f, φ) la distance des fonctions (f, φ), on a :

$$(\varphi, f) = (f, \varphi) \geq 0 ;$$
$$(f, \varphi) \leq (f, \psi) + (\psi, f)$$

quelles que soient les fonctions f, φ, ψ. Enfin on est amené dans le dèveloppement de la thèorie des fonctions de carrés intégrables à ne pas considérer comme distinctes deux fonctions dont la distance est nulle. (En fait, si $f(x)$ et $\varphi(x)$ sont continues sur x_0, x_1, leur distance ne peut être nulles que si $f(x) = \varphi(x)$ en tout point de x_0 x_1.)

Cette extension de la notion de distance permettait de généraliser immédiatement la notion de point-limite et de point d'accumulation.

5. *Classes* (D).—Une classe d'éléments de *nature quelconque* sera dite *classe* (D) si à tout couple d'éléments a, b de la classe on peut faire correspondre un nombre (a, b) que l'on appelle *distance* de ces deux éléments et qui jouit des propriétés suivantes :—

I $(a, b) = (b, a) \geq 0$

II deux éléments dont la distance est nulle ne sont pas considérés comme distincts.

III $(a, b) \leq (a, c) + (c, b)$ que les que soient les éléments a, b, c de la classe.

IV une suite infinie d'éléments a_1, a_2, ... est dite *convergente* s'il existe un élément a dont la distance à a_n peut devenir et rester aussi petite que l'on veut à partir d'un rang n assez élevé.

Et alors a est dit l'*élément-limite* de la suite convergente. (On voit en effet d'après II et III qu' une suite convergente ne peut avoir deux éléments-limites distincts.)

Ceci étant, un élément a sera dit *élément d'accumulation* d'un ensemble E d' éléments de la classe si l'on peut extraire de E une suite convergente dont a est élément limite.

6. Il est important de remarquer le rôle essentiel de la condition IV.

Tantôt en effet comme dans le cas des fonctions de carrés intégrables certaines analogies avec l'espace euclidien ont conduit à introduire un certain nombre appelé distance de deux élèments et alors la condition IV constitue proprement une définition des éléments limites. Tantôt au contraire certaines propriétés des éléments d' une certaine nature ont conduit à adopter a priori une certaine définition de la limite et il se trouve que cette définition est telle qu'on peut imaginer une définition de la distance satisfaisant aux conditions I, II, III et telle que la définition posée d'avance, des éléments limites satisfait à la condition IV.

Par exemple on a été amené dans certaines recherches à considèrer comme éléments des suites infinies de nombres réels ; on peut ainsi appeler point de l'espace à une infinité de coordonnées une suite infinie X de nombres réels

$$x_1,\ x_2,\ x_3,\dots.x_n\dots.$$

qu'on appellera les " coordonnées " du point X.

Suivant les questions traitées, différentes définitions des points-limites dans cet espace peuvent être données. On peut appeler *espace* (E_ω), l'espace dans lequel une suite de points

$$X^{(1)},\ X^{(2)},\dots.X^{(n)},\dots.$$

est dite converger vers le point-limite X si, quel que soit P les coordonnées de rang P

$$x^{(1)}_P,\ x^{(2)}_P,\dots.x^{(n)}_P,\dots.$$

de ces points tendent vers la coordonnée x_P de même rang P du point X. C' est la définitions de la limite qui s' impose à l'esprit la première et qui a été par exemple considérée par Hilbert sous le nom de convergence faible.

Or il est important de remarquer que l'espace E_ω est une classe (D), c' est à dire qu'on peut y définir une distance satisfaisant aux conditions I, II, III et telle que la définition des éléments limites, posée d'avance dans E_ω vérifie la condition IV. Il suffit en effet, d'appèler distance de deux points X, Y dont les coordonnées respectives sont

$$x_1,\ x_2,\dots.x_n\dots.$$
$$y_1,\ y_2,\dots.y_n,\dots.$$

la somme de la série convergente

$$(X,\ Y)=\frac{|x_1-y_1|}{1+|x_1-y_1|}+\frac{1}{2\,!}\,\frac{|x_2-y_2|}{1+|x_2-y_2|}+\dots.\frac{1}{n\,!}\,\frac{|x_n-y_n|}{1+|x_n-y_n|}+\dots.$$

7. La possibilité d'une définition de la distance satisfaisant à ces quatre conditions est, dans le cas actuel, la seule chose intéressante. L' expression formelle de cette distance est tout à fait secondaire, puisqu' une telle expression étant possible, on peut en déduire aussitôt une infinité d'autres qui n'altèrereraient ni la convergence des suites de points ni leurs limites éventuelles. (Par exemple (a, b) ètant la distance de deux éléments, on peut la remplacer par 2 (a, b) ou par

$$\left.\frac{(a, b)}{1 + (a, b)}\cdot\right)$$

8. Il faut bien insister sur ce que, ce qui caractérise les classes (D) ce n'est pas qu'on puisse y définir une distance satisfaisant aux conditions I, II, III, IV ; c'est que, une définition des éléments limites étant donnée d'une part, une définition de la distance satisfaisant à I, II, III étant donnée d'autre part, ces deux définitions soient entre elles dans une dépendance exprimée par IV.

On peut toujours en effet *pour n'importe quelle classe d'éléments* définir de bien des façons une *distance* satisfaisant à I, II, III et en déduire par IV une définition des éléments limites. Par exemple, on pourrait prendre $(a, b) = 1$ si a, b sont distincts et $(a, b)=0$ dans le cas contraire. I, II, III seraient vérifiés et il n'y aurait aucune suite convergente d'éléments distincts. Ou on pourrait, pour introduire des éléments limites, établir une correspondance quelconque entre une partie des éléments de la classe et les points (ou des points) d'une intervalle, et appeler distance de deux éléments la distance des deux points correspondants lorsqu'il y en a deux et la longueur de l'intervalle dans le cas contraire. En somme pour des éléments de nature déterminée et dans une théorie déterminée, les définitions de la distance et des éléments-limites, quoique théoriquement arbitraires, doivent, pour être utiles, être choisies convenablement, correspondre à la nature des choses. Et alors savoir si ces deux définitions sont liées par la relation•IV constitue un renseignement précieux.

9. Il faut en effet bien remarquer qu'il n'est pas toujours possible pour une définition donnée des éléments limites de choisir une définition de la distance liée à la précédente par la condition IV. Et ceci même sans forger artificiellement la définition des éléments-limites. Par exemple si l'on considère la classe des fonctions d'une variable réelle et si on y définit la limite d'une suite de fonctions $f_1(x)$, $f_2(x)$,.... sur un même intervalle $a\ b$, de la façon ordinaire, on peut prouver

(XXIX)* qu'il n'existe aucune définition de la distance satisfaisant dans cette classe aux conditions I, II, III, IV. C' est probablement l'impossibilité de donner une telle définition de la distance qui est la vraie raison du peu d'utilité de cette définition de la limite et qui explique pourquoi l'on doit toujours substituer dans la pratique à la convergence ordinaire, la convergence uniforme ou toute autre espèce comparable de convergence.

10. *Classes (D) complètes.* -Remarquons que dans toute classe (*D*) si une suite d'éléments $a_1, a_2, \ldots$ est convergente, on a en appelant a l'élément-limite :

$$(a_n, a_{n+p}) \leq (a_n, a) + (a_{n+p}, a)$$

Or à partir d'un certain rang (a_n, a) peut être rendu aussi petit que l'on veut. Donc pour n assez grand, on a, quelque soit p

$$(a_n, a_{n+p}) < \epsilon$$

ϵ étant un nombre positif donné d'avance. C'est là une propriété d'une suite convergente qui ne fait intervenir que ses propres éléments. Inversement si une suite d'éléments d'une classe (*D*) possède cette propriété *et si* cette suite a au moins un élément d'accumulation, la suite est convergente et a pour élément limite cet élément d'accumulation.

On voit que la réciproque comporte une restriction. Et en effet on peut citer des classes pour lesquelles une suite d'éléments peut posséder la même propriété sans être convergente. Et même, c'est le cas de la classe qui s'est présentée la première dans l'ordre historique. Lors que la notion de nombre a commencé à se préciser, les seuls nombres considérés étaient les nombres rationnels. Dans la classe (*D*) formée par les nombres rationnels *considérés exclusivement* et ou l'on appelle distance de deux de ces nombres la valeur absolue de leur différence, la suite des nombres formée par les racines carrés de 2 approchées à $\frac{1}{10^n}$ près n'est pas convergente et pourtant elle possède la propriété mentionnée si-dessus.

C'est seulement parce que cette classe a été complétée par l'adjonction d'éléments "impropres" ou "idéaux" à savoir les nombres irrationnels, que Cauchy a pu démontrer son théorème bien connu sur la convergence, pour la classe constituée par tous les nombres réels.

D'ailleurs remarquons bien que dans une classe (*D*), une même définition des éléments d'accumulation peut être obtenue par l'intermédiaire de deux définitions de la distance, l'une admettant une généralisation du thèorème de Cauchy et l'autre non. Par exemple, considérons la classe formée de tous les points d'un droite. On obtiendra la même définition de ses éléments d'accumulation si l'on passe par l'intermédiaire d'une distance (a, b), qu'on prenne pour (a, b) le segment (a, b) en valeur absolue ou qu'on prenne pour (a, b) l'arc (en valeur absolue) correspondant à $a\ b$ dans une inversion déterminée. La seconde définition n'est pas propre à la généralisation du thèoréme de Cauchy (quand on prend une suite de points s'éloignant à l'infini), la première l'est.

Nous appellerons classe (D) *compléte* ou classe (D_c), une classe (D) où l'une au moins des définitions de la distance qui n'altèrent pas la définition des éléments d'accumulation, admet une généralisation du théorème de Cauchy.

L' exemple donné plus haut, de la classe des nombres rationnels, montre qu'on peut parfois rendre une classe compléte par l'adjonction d'éléments nouveaux (ici les nombres irrationnels). Le second exemple le montre aussi (par la notion du point à l'infini) mais il montre également que cela n'est pas indispensable et qu' un simple changement de définition de la distance, changement qui n'altère ni l'étendue de la classe, ni la définition de ses éléments d'accumulation peut suffire. Or si l'adjonction des éléments impropres permet souvent de simplifier certains énoncés, certaines démonstrations, il éloigne généralement des applications. Considérer par exemple un prisme comme une pyramide (dont le sommet est à l'infini) a certainement quelques avantages théoriques mais c'est aussi ignorer une distinction essentielle dans le maniement des instruments employés à représenter graphiquement ces deux surfaces. Il ne faut introduire et surtout utiliser les éléments impropres qu'avec discrètion.

11. *Ecart.*—Bien que pour certaines définitions très classiques de la limite, il soit impossible de vérifier à la fois les conditions I, II, III, IV, c'est à dire de définir les éléments-limites par l'introduction d'une définition convenable de la "distance" il peut arriver qu'on ne soit pas obligé de renoncer à la fois à ces quatre conditions.

C'est en particulier ce qui a lieu quand on peut définir les éléments-limites par l'intermédiaire d'un "écart," c'est à dire en attachant à tout couple a, b d'éléments un nombre (a, b) appelé *écart* de a et de b, qui sans nécessairement satisfaire quels que soient a, b, c à la condition

$$\text{III} \qquad (a, b) \leqq (a, c) + (c, b)$$

satisfasse à la fois aux conditions I, II, IV, du no. 5. Dans ce cas nous appelleros la classe *une classe "E."* L'introduction d'un terme spécial est justifié par l'existence de certaines classes (voir par exemple le paragraphe suivant) où la définition de la limite est compatible avec une définition d'un écart satisfaisant aux conditions I, II, IV sans qu'il soit possible de modifier la définition de cet écart de fàçonà le faire vérifier la condition III sans altérer la convergence de ses suites d'éléments ou leurs limites. La notion d'écart est donc plus générale que cebe de distance. Il serait d'ailleurs intéressant de donner un exemple moins artificiel que celui du paragraphe suivant.

Classes (E_c).—On peut définir les classes (E) *complétes* ou *classes* (E_c) de la même façon que nons avons défini les classes (D) complétes ou (D_c). Seulement, dans ce cas, la condition de Cauchy non seulement n'est pas suffisante pour assurer la convergence, mais on ne peut plus s'appuyer sur la condition III pour montrer qu'elle est nécessaire. On pourrait se demander s'il existe des classes (E) qui soient complétes sans être des classes (D). La réponse est affirmative comme le montre l'exemple suivant.

Considérons la classe formée des points du plan

a_0 de coordonnées $(0, 0)$

a_n de coordonnées $\left(\frac{1}{n}, 0\right)$; $n = 1, 2, \ldots.$

$a_n^{(p)}$ de coordonnées $\left(\frac{1}{n}, \frac{1}{p}\right)$; $n = 1, 2, \ldots$; $p = 1, 2, \ldots.$

Et définissons y pour écart de deux de ces points leur distance géométrique quand ils sont tous deux sur ox ou tous deux sur une même parallèle à oy et un écart égal à l'unité dans le cas contraire. Définissons-y les suites convergentes et leurs éléments limites par l'intermédiaire de cet écart. Alors une suite convergente devra à partir d'un certain rang ne contenir que des points a_n ou seulement des points $a_n^{(p)}$ correspondant à une même valeur de p. Alors on voit bien que la classe sera une classe (E) compléte. Cependant on ne peut d'aucune manière y définir une distance satisfaisant à la condition III. XI, page 55, § (6).*

11$^{\text{bis}}$. *Anciennes classes* (V).—Les classes (D) sont des classes (E) dans lesquelles, en particulier, la condition (voir § 18).

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* Les chiffresromains renvoient à la liste bibliographique placée à la fin de ce mémoire (§ 65).

5° Tout ensemble dérivé est fermé

est vérifiée. Si on sait seulement d'une classe (E) qu'elle vérifie la condition 5°, on peut démontrer (XI, page 63) qu' à tout élément b et à tout nombre positif δ on peut faire correspondre un nombre η tel que si c est un élément déterminé dont l'écart avec b soit moindre que η et si ω est un nombre convenablement choisi une fois connus δ, η, b, c, alors pour tout élément d dont l'écart avec c est moindre que ω, on aura $(b, d) \angle \delta$.

Dans le cas où en outre ω ne dépend pas de b, il existe une fonction $\psi(\theta, b)$ telle que

$$(b, d) \underline{\angle} \psi((b, c) + (c, d), b)$$

$\psi(\theta, b)$ tendant en décroissant vers zèro quand le nombre θ tend en décroissant vers zéro Dans le cas où en outre $\psi(\theta, b)$ est indépendant de l'élément b, on voit qu'en posant $\varphi(\epsilon) = \psi(2\epsilon)$, les inégalités

$$(1) \qquad (b, c) \angle \epsilon, \ (c, d) \angle \epsilon,$$

entraînent

$$(2) \qquad (b, d) \angle \phi(\epsilon)$$

$\phi(\epsilon)$ étant une fonction qui tend vers zéro avec ϵ

J'avais été amené dans ma Thèse à introduire à côté des classes (D) (que j'appelais alors classes (E)), les classes que j'appelais alors classes (V) et où un écart (que j'appelais alors voisinage) peut se définir de façon à satisfaire à la dernière condition exprimée par (2) quand (1) est vérifié. J'avais cru alors généraliser utilement la notion de classe (D). Mais j'eus bientôt la perception que cette généralité était plus apparente que réelle (XIX, § 33, p. 22). Cette conviction a été justifiée par E.W. Chittenden qui a démontré (XXIII) que dans toutes les classes que j'appelai (V) dans ma Thèse, on peut sans changer les éléments d'accumulation définir une distance satisfaisant à la condition IV du § 5 : autrement dit ces anciennes classes (V) sont toutes des classes (D). Il n'y a donc plus lieu de leur faire un sort à part et c'est pourquoi on ne les trouvera pas mentionnées dans la Seconde Partie de ce mémoire.

12. *Classe* (L).—Mais si l'on veut réduire la notion de limite à ses éléments essentiels, on doit encore aller plus loin. En effet, nous avons rappelé au § 9 qu'on ne peut déduire la convergence ordinaire des suites de fonctions d'une variable d'aucune définition de la distance satisfaisant aux conditions I, II, III, IV. Or, il en est de même lorsqu' on laisse de coté la condition III (XI, page 59, § 11) : On ne peut déduire la convergence ordinaire des suites de fonctions d'une variable d'aucune définition de l'écart de deux fonctions.

Il y a donc lieu de considérer d'une maniére générale les *classes* (L) où la définition des suites convergentes est donnée a priori indépendamment de toute notion d'écart entre deux éléments de la classe. Mais pour nous conformer à l'intuition que nous avons de la notion de limite et en particulier pour adopter des conditions vérifiées par la convergence ordinaire des suites de fonctions, qui soient indépendantes de la nature des éléments considérés, nous supposerons que les deux conditions suivantes soient remplies.

Nous avons admis que pour toute suite donnée d'avance $a_1, a_2, a_3 \ldots$ d'éléments de la classe la définition de la limite (que nous n'avons pas à préciser) aie pour effet de classer cette suite comme divergente ou comme convergente et dans ce second cas de lui attacher un élément unique bien déterminé appelé élément-limite de la suite. Mais cette définition doiê tre telle que :

1° si les éléments $a_1, a_2, \ldots$ sont identiques au premier la suite est classée comme convergente et a_1 est son élément-limite.

2° les éléments $a_1, a_2, \ldots$ étant distincts ou non si leur suite est convergente, toute suite infinie extraite de celle ci est aussi convergente et a le même élément-limite.

Ce sont là les deux conditions essentielles. Mais il faut remarquer que si l'on n'a en vue que la détermination des éléments d'accumulation, on pourrait (X, page 11, § XIII) sans changer ceux-ci modifier au besoin une définition donnée des suites convergentes et des éléments limites satisfaisant aux deux conditions prècèdentes, de sorte que la nouvelle définition satisfasse encore à ces deux conditions et en outre aux suivantes :

3° si une suite S converge vers un élément a, il en est de même de toute suite obtenue en ajoutant à S un nombre fini d'éléments (distincts ou non).

4° si un nombre fini de suites $S_1, S_2, \ldots S_n$ convergent vers a il en est de même de toute suite obtenue en rangeant en une seule suite les éléments (distincts ou non) de $S_1, S_2, \ldots S_n$.

Bien entendu, toute classe (D) et même toute classe (E) est une classe (L). Il suffit d'y définir une suite convergente $a_1, a_2, \ldots$ et son élément limite a comme telles que l'écart (a_n, a) tend vers zéro avec $\frac{1}{n}$.

13. *Nouvelle extension de la notion de limite.*—Mais la considération des classes (L) ne peut être considérée comme le dernier terme de l'évolution de la notion d'élément d'accumulation. C'est ainsi que pour donner une démontration plus simple du théoréme de Cantor-Bendixson Lindelöf a été amené à introduire la notion de point de condensation d'un ensemble linéaire E. Il appelle ainsi tout point x tel qu'il existe une infinité non dénombrable de points de E dans tout intervalle de centre x. Or rien ne nous empêche de considérer une classe nouvelle formée des points d'une droite, mais où tout point d'accumulation serait défini comme Lindelöf définit les points de condensation. La correspondance qui détermine les éléments d'accumulation, au sens nouveau, d'un ensemble linéaire est parfaitement définie. Cependant elle est définie sans passer par l'intermédiaire de la définition des suites dénombrables convergentes. Il est même impossible dans le cas actuel de passer par cet intermédiaire ; en d'autres termes, on ne peut trouver aucune définition équivalente de ces points d'accumulation qui permette de considérer cette nouvelle classe comme une classe (L). En effet dans cette nouvelle classe aucun ensemble dénombrable ne saurait avoir d'élément d'accumulation au sens nouveau. (Nous verrons pourtant plus loin que cette classe satisfait aux conditions 1°, 2°, 3°, 4°, 5°, 7°, des § 15, 16).

14. *Introduction du voisinage.*—En introduisant la considération des éléments d'accumulation comme dépendant essentiellement des notions de suites convergentes et d'éléments-limites nous avons suivi l'ordre historique. Mais la notion d'élément d'accumulation a déja débordé son cadre original. Tel que nous l'avons d'abord défini, un élément d'accumulation d'un ensemble doit être en même temps élément d'accumulation d'une suite dénombrable particulière d'éléments de l'ensemble. Les progrès de l'analyse ont conduit à une conception plus générale de la limite, comme nous venons de le voir. Avant d'arriver au cas général, l'exemple suivant nous feu franchir un nouveau stade.

Considérons par exemple une suite *ordonnée* d'éléments, c'est à dire une ensemble d'éléments tels que deux d'entre eux étant donnés' une loi déterminée fixe celui des deux qui est dit d'un rang inférieur à l'autre ce qu'on exprime par la notation

$$a \angle b$$

avec la condition que $a \angle b$ et $b \angle c$ donnent $a \angle c$. On pourra alors appeler segment $a\, b$ l'ensemble des éléments α tels que $a \angle \alpha \angle b$ Et on dira qu'un élément a est un élément d'accumulation d'un certain ensemble E d'éléments de cette suite ordonnée si tout segment auquel appartient a a aussi un élément en commun autre que a avec E. Dans le cas où la suite ordonnée est composée des points d'une droite rangés dans leur ordre géométrique cette définition est équivalente à la définition par l'intermédiaire des suites dénombrables convergentes. Mais dans le cas général, il peut trés bien arriver qu' aucune suite dénombrable n'ait d'éléments limites.

15. *Classes* (V).—Cette définition attribue un rôle fondamental dans la détermination des éléments d'accumulation à des ensembles partiels attachés à chaque élément de la classe et qui jouent le rôle de voisinages de cet élément.

Mais rien n'oblige à limiter la conception de voisinage à l'exemple précédent. C'est ainsi qu'en 1911, Ralph Root (XII) considère le cas de voisinages quelconques assujettis seulement à quelques conditions qui n'en déterminent pas la nature et qu'il précise dans un mémoire de 1914. C'est ainsi qu'en 1912, F. Hausdorff développe oralement dans son cours et qu'en 1914, il expose dans son livre "Grundzüge der Mengenlehre" (XXV, pages 211, 213) la théorie de *l'espace topologique* fondée sur des considérations analogues. Mais je ne donnerai pas ici la définition de cet espace que nous retrouverons plus loin comme cas particulier de considérations plus générales.

Rien n'empêche en effet de partir de la notion la plus générale de voisinage et de ne lui imposer des conditions restrictives qu'en vue d'applications déterminées. C'est après avoir présenté dans ce sens deux généralisations de plus en plus étendues parues en 1906 (XVIII, pages 17, 18) et en janvier 1918 (XI, page 57), que j'ai publié en mai 1918 au moyen de notes esquissées depuis longtemps et mises au point au front, un mémoire où j'expose ce point de vue (X, page 3).

On peut donc présenter la notion d'élément d'accumulation de la façon suivante. A chaque élément a d'une certaine classe, on attache une certaine famille d'ensembles V_a appelés *voisinages* de a. On dit alors que a est élément d'accumulation d'un ensemble E si E comprend des éléments aussi voisins que l'on veut de a, c'est à dirè si tout voisinage V_a de a, a avec E un élément au moins, autre que a, en commun.

Nous appellerons *classe* (V) toute classe d'éléments où les éléments d'accumulation sont définis par l'intermédiaire de *voisinages* de la manière que nous venons d'indiquer.

Il est évident que les seuls éléments des voisinages V_a qui interviennent dans cette définition sont distincts de a ; il est donc indifférent d'ajonter a à V_a ou de l'en retrancher. Pour éviter toute confusion, nous supposerons toujours sauf mention du contraire que V_a contient a.

Il est évident également que si l'on remplace une famille de voisinages de a par une autre on n'altère pas nécessairement la collection des ensembles dont a est élément d'accumulation. Il est alors utile de préciser à quelles conditions deux *familles* $\{V_a\}$, $\{W_a\}$ de voisinages de a sont *équivalentes* à ce point de vue : il faut et il suffit que tout voisinage V_a contienne l'un des voisinages W_a et réciproquement (X, page 5, § VIII).

Exemples de classes (V). Toute classe (D) et même toute classe (E) peut être considérée comme une classe (V). Il suffit de choisir comme voisinage d'un élément a, les sphéroïdes de centre a et de rayous quelconques, en appelant sphéroïde de centre a et de rayon ρ, l'ensemble des éléments b tels que $(a, b) \leqq \rho$. On obtient une famille équivalente en ne prenant pour ρ que les inverses des nombres entiers.

Une classe (L) peut être aussi considérée comme une classe (V). Il suffit de prendre comme voisinages d'un élément a les ensembles auxquels a est intérieur.

Même la classe considérée au § 13, où aucune suite dénombrable n'a d'élément d'accumulation peut être considérée comme une classe (V). C'est alors la classe formée des points d'une droite où l'on considére comme voisinages d'un point a les intervalles de centre a desquels on retire une suite dénombrable quelconque de points distincts de a.

16. Mais les considérations précédentes montrent que c'est la notion d'élément d'accumulation qui doit être primordiale. Il y a donc lieu de n'en limiter en rien la portée.

Nous pourrons alors *généraliser encore une fois* la notion de point limite de la façon suivante.

Nous supposerons donnée une certaines classe d'éléments de nature quelconque et une certaine *loi de correspondance* K qui attachera d'une manière d'ailleurs tout à fait quelconque à tout ensemble E d'éléments de la classe un certain ensemble E' d'éléments de la même classe, ensemble qu'on appellera *ensemble dérivé* de E. Le fait que E' est le dérivé de E pourra s'exprimer par la relation $E \; K \; E'$.

On appellera alors *élément d'accumulation* de E tout élément de son ensemble dérivé E' *.

17. *Les quatre conditions de* F. Riesz.—Suivant la nature de la question où intervient la classe considérée la relation K aura telle ou telle forme. Si l'on examine les différentes définitions des éléments d'accumulation qui ont été passées en revue jusqu'ici, on voit qu'elles satisfont à quatre conditions qui ont été explicitement énoncées pour la première fois par F. Riesz. (XXV, pages 19, 20).

Ces conditions sont les suivantes.

1° Tout élément d'accumulation d'un ensemble est aussi élément d'accumulation des ensembles dont le premier fait partie.

2° Si l'on partage un ensemble en deux parties, tout élément d'accumulation de l'ensemble est élément d'accumulation de l'une au moins de ces deux parties.

3° Un ensemble composé d'un seul élément, n'a aucun élément d'accumulation.

4° Un élément d'accumulation d'un ensemble E est déterminé par l'ensemble des sous-ensembles de E dont il est élément d'accumulation.

Ces conditions sont indépendantes entre elles. Si on en suppose plusieurs vérifiées en même temps, on peut les formuler d'une manière un peu différente.

1° et 2° : L'opération de la dérivation d'un ensemble est une opération distributive ; ce qui s'exprime par la notation symbolique

$$(E+F)'=E'+F'$$

appliquée à deux ensembles quelconque E, F.

3° (en tenant compte de 2°) : Un ensemble ne peut avoir d'éléments d'accumulation que s'il comprend un nombre infini d'éléments.

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* Dans un mémoire sous presse (XXXVIII) N. Wiener part d'un antre point de vue et qui peut être fécond. Il se donne un ensemble $\geq$ de transformations de la classe en ellemême et définit un élément d' accumulation A de E, comme un elément invariant dans toute transformation de E ani laisse invariant cheun des élements de E sauf peut-être A. Il y aurait lieu d'étudier directement le problème qui est l'origine et la raison d'être de la définition précédente :

On se donne un ensemble E de transformations de la classe ; la relation $E\ K\ E'$ sera telle que les transformations (supposées biunivoques) de $\geq$ soient aussi bicontinues ; on demande à quelles conditions doit satisfaire $\geq$ pour que se problémé soit possible et que la relation K soit détermenée.

4° Si a et b sont deux éléments d'accumulation de E qui sont distincts, il existe un sous ensemble de E dont l'ensemble dérivé comprend a et non pas b ou inversement.

18. *Autres conditions.*—On peut ranger à la suite de ces conditions de F. Riesz deux autres conditions indépendantes entre elles, indépendantes des conditions de F. Riesz et qui sont vérifiées par un grand nombre des classes que l'analyse amène à considérer, mais non pas par toutes :

Avant de les énoncer, faisons comprendre la nécessité de leur introduction. Si l'on considére un ensemble E d'éléments d'une classe (D), cet ensemble peut avoir des éléments d'accumulation qui formeront l'ensemble dérivé E'. Celui-ci à son tour peut avoir des éléments d'accumulation, formant l'ensemble dérivé du second ordre E''. Or si a appartient à E'' il y a des éléments (distincts de a) a_n de E' tels que $(a, a_n) < \frac{1}{n}$ et par suite des éléments b_n de E, distincts aussi de a, tels que $(a_n, b_n) < \frac{1}{n}$. Donc $(a, b_n) < \frac{1}{n}$, a appartient à E'. Autrement dit l'ensemble E' appartient à la catégorie des ensembles *fermés*, c'est-à-dire des ensembles qui contiennent leur ensemble dérivé. Une telle propriété n'est pas vérifiée par toutes les définitions classiques de la limite. Considérons par exemple la classe (L) formée par les fonctions uniformes d'une variable réelle sur un intervalle fixe I et définissons y la limite d'une suite de fonctions à la façon ordinaire, sans préoccupation d'uniformité de la convergence. Dans cette classe, l'ensemble E des fonctions continues a un certain ensemble dérivé E', l'ensemble des fonctions dites de premierè classe au sens de Baire. De même l'ensemble E'' dérivé de E' forme l'ensemble des fonctions de seconde classe au sens de Baire, etc... Or l'intérêt même de cette classification consiste en ce que E n'est qu'une partie de E', E' une partie de E'' etc... Par exemple, la fonction égale à un pour toute valeur rationnelle de la variable et nulle ailleurs appartient à E'' sansi appartenir à E'. Autrement dit E' n'est pas fermé. On est ains amené à ranger à côté des conditions de F. Riesz la condition :

5° *Tout ensemble dérivé est fermé :*

Cette condition est vérifiée dans toute classe (D) ; elle n'est pas vérifiée dans toute classe (L) comme le montre la classification de Baire. Inversement elle peut être vérifiée sans que les conditions de F. Riesz soient remplies. On peut citer (X, page 7) une classe qui n'est pas une classe (V) et vérifie pourtant 5°.

19. Pour arriver à l'introduction de la dernière condition annoncée, nous remarquerons l'importance que joue en mathématiques le fait que tout nombre irrationnel est la limite de nombres rationnels. Un grand nombre de propriétés se démontrent en les établissant pour le cas des nombres rationnels et passant ensuite à la limite (évaluation des volumes de la géometrie élémentaire, propriétés de l'exponentielle, etc.). Il peut arriver (comme dans les exemples cités) que la méthode réussisse par suite des propriétés arithmétiques des nombres rationnels. Mais il peut arriver que la seule propriété des nombres rationnels réellement utilisée soit le fait que leur ensemble est dénombrable. C'est par exemple le cas de la méthode de Borel pour établir une formule d'interpolation valable pour toute fonction continue. Nous voyons ainsi utilisée la propriété suivante du continu linéaire, celle d'après la quelle tout nombre réel appartient à un certain ensemble dénombrable (celui des nombres rationnels) ou à son ensemble dérivé. Or il importe de faire remarquer que si non seulement le continu linéaire, mais encore bien d'autres classes importantes possédent une propriété analogue cette propriété peut n'avoir pas lieu et ceci même pour des classes dont la considération se présente naturellement. Remarquons d'abord que si tout élément *d'une classe* (*L*) appartient à un certain ensemble dénombrable ou à son dérivé chaque élément de la classe pourrait être défini par une suite dénombrable d'indices entiers. Donc la classe aurait au plus la puissance du continu. Nous savons d'autre part que la classe considérée plus haut des fonctions d'une variable réelle a une puissance supérieure à celle du continu (I). On peut même citer une classe où on peut définir une distance et on cette propriété n'est pas vérifiée. C'est la classe dont chaque élément x est définipar une suite infinie de nombres réels x_1, $x_2 \ldots . x_n$ ..qui sont ses coordonnées, ces coordonnées étant supposées, pour un élément x déterminé, bornées quel que soit n ; classe où nous définissons la distance de deux éléments x, y comme la borne supérieure de la différence $/x_n - y_n/$ de leurs coordonnées de même rang (VI, page 161).

Il nous sera donc utile de distinguer d'un nom spécial cette propriété : nous dirons qu'un ensemble E est *séparable* quand il existe un ensemble dénombrable N d'éléments de cet ensemble tel que tout élément de E appartienne à N oùen soit un élément d'accumulation.

Le dernier exemple que nous venons de donner montre que les conditions 1°, 2°, 3°, 4°, 5° (et aussi la condition 7° énoncée plus loin) peuvent être vérifiées sans que la classe soit séparable. On pourrait aussi invoquer l'exemple donné au § 13.

L'exemple de la classe dénombrable considérée au § 11 montre qu'une classe peut être séparable et vérifier les quatre conditions de F. Riesz sans vérifier 5°.

Enfin en choisissant comme classe les nombres rationnels, on définit facilement les points d'accumulation de façon à avoir une classe non (V) mais séparable.

En définitive on voit qu'il est important de distinguer si une classe vérifie ou non la condition d'être séparable, c'est à dire la condition :

> 6° Il existe au moins un ensemble dénombrable N d'éléments tel que tout élément appartienne à N ou à son ensemble dérivé.

Enfin il est évident qu'une classe aura des propriétés plus homogènes si elle vérifie la condition.

> 7° Tout élément est élément d'accumulation d'au moins un ensemble d'éléments de la classe considérée.

Cette condition est elle même indèpendante de toutes les prècèdentes ; par exemple la classe (D) qui sera formée par une droite et un point pris en dehors de la droite en y adoptant la définition ordinaire de la distance ne vérifie pas 7° mais vérifie les six autres conditions.

20. *Remarque.*— Dans les questions de géométrie de situation, on serait amené à remplacer la condition 7° par une condition qui la comprend mais plus restrictive encore et exprimant que la classe forme un tout cohérent sous la forme suivante :

> 8° La classe est un continu :

Nous dirons avec Jordan qu'un *continu* est un ensemble fermé ; contenant plus d'un élément et qu'on ne peut diviser en deux ensembles fermés disjoints.* Par définition même une classe est un ensemble fermé. Elle perdrait tout intérêt si elle ne contenait qu'un élément.

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* Deux ensembles sont *disjoints* s' ils n'ont aucun élément commun.

La condition 8° peut donc se réduire à la suivante : la classe considérée ne peut être décomposée en deux ensembles fermés disjoints. Appelons ensemble *ouvert* un ensemble dont aucun élément n'est élément d'accumulation de l'ensemble complémentaire. On voit alors que la condition 8° peut encore s'ènoncer ainsi :

Un ensemble ne peut être à la fois ouvert et fermé.

Il est d'ailleurs évident sur la seconde forme de la condition 8° que celle-ci entraîne la condition 7°. La réciproque n'est pas vraie comme le montrerait l'exemple d'une classe formée des points d'un plan sauf un point de ce plan, la limite y étant définie comme d'ordinaire.

21. Les conditions 1° à 8° sont évidemment compatibles puis qu'elles sont réalisées dans le cas des ensembles linéaires. En ce qui concerne leur interdépendance nous savons déja que 8° entraine 7°, laissons donc de coté pour le moment la condition 8°.

Les conditions 1° à 7° sont indépendantes. Il suffit de donner des exemples de classes vérifiant toutes ces conditions sauf l'une d'entre elles.

On peut vérifier 2° à 6° sans vérifier 1° en prenant pour classe l'ensemble des points d'une droite D et d'un point extérieur a et en prenant comme ensemble dérivé d'un ensemble E l'ensemble dérivé ordinaire quand E ne comprend pas a et toute la droite D quand E comprend a et une infinité de points de D, l'ensemble dérivé de a seul étant nul.

L'exemple des ensembles linéaires montre d'ailleurs que les conditions 1° à 8° sont compatibles.

22. En combinant entre elles un nombre plus ou moins grand de ces conditions, on obtient des classes moins ou plus générales et dans lesquelles on pourra généraliser un nombre plus ou moins grand des propriétés des ensembles linéaires. Nous allons passer en revue quelques unes de ces classes.

23. *Classes* (V).—On voit évidemment que les classes (V), définies au § 13 vérifient la condition 1° de F. Riesz (§ 14) et une condition qui est une conséquence de l'ensemble des conditions 2°, 3° de F. Riesz sans être équivalente à cet ensemble :

(b) si un élément a appartient à la fois à un ensemble F et à son dérivé F', il reste aussi élément d'accumulation de F quand on supprime a de F.

L'ensemble des conditions 1° et (*b*) est équivalent à l'ensemble de la condition 1° et de la condition suivante :

la propriété pour un ensemble E d'avoir pour élément d'accumulation a ne dépend que des éléments de E distincts de a.

D'autre part, réciproquement si dans une classe la relation $E\ K\ E'$ entre les ensembles et leurs dérivés vérifie les conditions 1° et b, cette classe est (V) et on peut choisir X comme famille de voisinages de chaque élément a, la famille constituée par les ensembles auxquels a est ***intérieur***. (On dit que a est intérieur à E, si a appartient à E et si a n'est pas élément d'accumulation de l'ensemble des éléments n'appartenant pas à E).

Bien entendu, il peut arriver que, les familles de voisinages étant données d'avance, la famille relative à un élément a ne comprenne pas tous les ensembles auxquels a est intérieur. Mais quelle que soit cette famille, a sera intérieur à tous ses voisinages et de plus pour que a soit intérieur à un ensemble E il faut et il suffit que l'un au moins des voisinages de a appartienne tout entier à E.

24. Il est intéressant de rechercher à quelles conditions doivent satisfaire les familles de voisinages, jusqu'ici entièrement arbitraires pour que l'une ou l'autre des conditions de 1° à 7° soit vérifiée.

La condition 1° est vérifiée d'elle même dans toute classe (V).

Pour une classe (V) les conditions 2° à 7° sont respectivement équivalentes (X, pages 6, 7) aux suivantes.

2° bis. L'ensemble commun à deux voisinages quelconques de tout élément a contient entièrement au moins un des voisinages de a.

3° bis Les voisinages d'un élément quelconque a n'ont en commun que le seul élément a.

4° bis Pour tout couple d'éléments distincts a et b il existe au moins un voisinage de a qui ne comprend entiérement aucun voisinage de b et inversement. (Tout au moins ceci doit avoir lieu quand on retranche préalablement des voisinages de chaque élément cet élément). C'est la condition qu'on obtient quand dans la condition 4° on prend pour E la classe elle-même. Si l'on veut que la condition soit vérifiée pour tout ensemble E ayant a et b comme élément d'accumulation, il faudra que la condition équivalente énoncée plus haut soit encore vérifiée quand on remplace les voisinages par leur partie commune avec E.

5° bis Quels que soientl 'élément a et le voisinage V_a de a il existe un voisinage V_a° de a dont tout élément b posséde au moins un voisinage V_b° appartenant entièrement à V_a.

6° bis Il existe un ensemble dénombrable N d'éléments tel que tout voisinage V_a de tout élément a contienne un élément au moins de N (élément qui pourra être distinct ou non de a).

7° bis Tout voisinage possède au moins deux éléments distincts.

Ces nouvelles conditions donnent le moyen de montrer, parfois plus commodément, que les conditions de F. Riesz sont indépendantes. Par exemple on voit facilement que la classe des points d'une droite est une classe (V) qui vérifie les conditions 1°, 3°, 4°, 7° sans vérifier la condition 2°, ni 5°, ni 6° lorsqu'on adopte comme famille de voisinages de chaque point x, la famille des suites de points qui convergent au sens ordinaire vers x. Autrement dit dans cette classe un point x serait point d'accumulation d'un ensemble E s'il était le centre d'un intervalle ne comprenant, sauf peut être x, que des points de E. Ainsi 2° est indépendante de 1°, 3°, 4°.

25. Dans le cas où plusieurs des conditions 1° à 7° sont vérifiées, les conditions équivalentes formulées pour les classe (V) peuvent se simplifier.

Par exemple si 2° et 3° sont déja vérifiées, on peut supprimer de 4° bis la restriction entre parenthèses.

Si 2° est vérifiée, la condition 5° bis peut s'énoncer ainsi (X, page 9).

5° ter : pour tout élément a et tout voisinage V_a de a, il existe un voisinage V_a' de a dont tous les éléments sont intérieurs à V_a.

Si 2° est vérifiée, la condition 8° s'exprime ainsi pour une classe (V) :

> 8° bis même en remplaçant les familles de voisinages par des familles équivalentes, il est impossible d'effectuer une coupure dans la collection des voisinages de tous les éléments, c'est-à-dire de les séparer en deux catégories de sorte qu'un voisinage quelconque de la première catégorie soit disjoint d'un voisinage quelconque de la seconde catégorie.

26. *Classes* (*H*).—Entre les classes (V) d'une généralité extrême et les classes (*D*) qui se rapprochent de l'espace euclidien, il est utile de considérer une catégorie presque aussi simple que les classes (V) et qui

jouissent d'un grand nombre de propriétés importantes, comme on le verra plus loin. Il s'agit des *classes* (*H*) *qui vérifient les conditions* 1°, 2°, 3°, 5°. En cherchant à généraliser certaines propriétés des ensembles linéaires, on s'aperçoit en effet que la condition 4° de F. Riesz joue un role beaucoup moins important que la condition 5°. Une telle classe peut être encore définie de la façon suivante.

C'est une classe (V) dont les voisinages vérifient les conditions 2° bis, 3° bis, 5° bis du § 24. On peut présenter celles-ci d'une façon légérement différente :

On remarque (XXXII, § 23) que dans une classe (H), on obtient une famille $\{W_a\}$ de voisinages équivalente à une famille donnée $\{V_a\}$ de voisinages de a en prenant pour W_a l'intérieur de V_a. Alors chaque voisinage W_a est un ensemble *ouvert* et la classe (H) est finalement une classe (V) satisfaisant aux conditions suivantes :

(*A*) A tout élément b correspond au moins un voisinage W_b; chaque voisinage W_b contient l'élément b.

(*B*) Etant donnés deux voisinages W_b^{0}, W_b' de b, il existe un voisinage W_b^{2} de b appartenant à la fois à W_b^{0} et à W_b'.

(*C*) Quel que soit l'élément c du voisinage W_b de b, il existe un voisinage W_c de c qui appartient entièrement à W_b.

3° bis Pour tout couple d'éléments distincts b, c, il existe un voisinage W_b de b ne comprenant pas c et inversement.

On reconnait dans les conditions (A), (B), (C) les trois premières conditions imposées par F. Hausdorff (XIV, page 213) à son "espace topologique." Il impose en outre au lieu de 3° bis la condition

(*D*) Pour tout couple d'éléments distincts b, c, il existe deux voisinages respectifs de b et de c qui sont disjoints.

Or il faut remarquer que cette condition entraîne *les deux* conditions 3° bis et 4° bis du § 24. Par conséquent *l'espace topologique de F. Hausdorff est une classe* (*H*) et la classe (H) est plus générale. Or on verra plus loin que les propriétés les plus importantes des classes (D) qui se généralisent à l'espace topologique (une classe (D) est évidemment un espace topologique) se généralisent au delà jusqu'aux classes (H)

27. *Ensembles bornés.*—Dans la théorie des ensembles linéaires, on n'attache pas une grande importance à la condition pour un ensemble d'être borné, c'est à dire contenu dans un intervalle fini. Il est en effet si facile de reconnaître qu'un ensemble est borné et de

distinguer les propriétés des ensembles bornés qui ne s'étendent pas aux ensembles illimités, qn'on néglige assez souvent de spécifier si les ensembles dont on parle sont bornés ou non. Bien que le danger d'une confusion soit petit, il existe pourtant. Il grandit dans le cas d'ensembles plans où par exemple les définitions d'un continu dues à Cantor et à Jordan, équivalentes dans le cas de continus bornés sont nettement distinctes dans le cas général.

Mais dès qu'on étudie des classes d'éléments d'une nature plus générale comme la classe des fonctions continues, la nécessité de distinguer entre les ensembles qui généralisent les ensembles linéaires bornés, et les autres devient absolument essentielle. Une difficulté se présente immédiatement : à quoi reconnaîtra-t-on qu'un ensemble est borné Pour certaines classes, la généralisation parait immédiate et elle l'est en effet. Par exemple, on peut appeler ensemble borné de fonctions continues un ensemble de fonctions continues dont les valeurs absolues restent inférieures à un nombre fixe. On appellera ensemble borné de courbes continues un ensemble de courbes continues, toutes situées dans une sphère fixe, etc.....

Mais si l'on veut étendre la notion d'ensemble borné à un ensemble abstrait des généralisations de ce genre, essentiellement fondées sur la nature particulière des éléments considérés ne sont plus possibles. Tout au plus peut-on appeler ensemble borné d'éléments d'une classe (D) [on d'une classe (E)] un ensemble d'éléments contenus dans un sphéroïde fini. Et encore peut-il arriver, comme c'est le cas pour l'espace E_ω, que la définition ainsi obtenue ne coïncide pas avec la définition qui découle naturellement de la nature des éléments considérés. Mais la plus grande difficulté n'est pas là.

La difficulté est non pas de généraliser d'une façon naturelle la définition des ensembles linéaires limités, mais de la généraliser de façon utile. Autrement dit, il faut généraliser la notion d'ensembles linéaires limités de telle manière que le plus grand nombre possible des propriétés de ces ensembles puissent être aussi généralises. On s'aperçoit alors qu'il faut préalablement substituer à la définition des ensembles linéaires bornés une définition équivalente mais exprimée sous une forme ne faisant pas intervenir la nature particulière des ensembles linéaires.

28. *Ensembles compacts et parfaitement compacts.*—On peut faire cette substitution de plusieurs manières. Celle que j'ai indiquée dans

ma Thèse (XVIII, page 6, § 9) a été choisie pour s'adapter au cas des classes (L) qui en était l'objet Elle est fondée sur le théorème de Weierstrass-Bolzano, d'après lequel tout ensemble infini (c-a-d comprenant une infinité d'éléments distincts) linéaire borné a au moins un élément d'accumulation. Pour en tirer une nouvelle définition des ensembles linéaires bornés il faut le compléter sous la forme suivante : la condition nécessaire et suffisante pour qu'un ensemble linéaire *E* soit borné est que : tout sous ensemble infini de *E* admet un élément d'accumulation. On voit alors immédiatement comment on poura généraliser la notion d'ensemble borné. Mais pour distinguer des généralisations en apparence plus naturelles que nous avons mentionnées plus haut, j'ai introduit une nouvelle désignation :

Nous appellerons donc *ensemble compact* un ensemble *E* tel que tout sous-ensemble infini *F*, de *E* admet au moins un élément d'accumulation. Cet élément n'appartient nécessairement ni à *F*, ni à *E*. Lorsqu'il existe toujours au moins un élément d'accumulation de *F* appartenant à *E* quel que soit le sous ensemble infini *F* de *E*, on dira que *E* est *compact en soi* suivant une dénomination que j'emprunte à Chittenden.

Non seulement cette nouvelle définition coincide avec celle des ensembles bornés quand ceux-ci sont linéaires; non seulement elle garde un sens précis pour des classes (L) et même pour des classes où le choix des éléments d'accumulation n'est soumis à aucune restriction ; mais surtout elle permet comme on le verra par la suite de généraliser très loin les principales propriétés des ensembles linéaires bornés.

Cependant R. L. Moore a récemment remarqué (XVII) qu'on peut encore gagner en généralité en lui substituant une définition qui est en général plus restrictive, mais qui est équivalente à celle des ensembles compacts dans le cas très général des classes (D). Employant une locution employée dans un autre cas par S. Janiszewski, nous présenterons la définition de R. L. Moore de la façon suivante :

Nous appellerons d'abord avec R. L. Moore *suite monotone* d'ensembles une collection d'ensembles tels que de deux d'entre eux l'un soit toujours une partie de l'autre.

On dira qu'un ensemble *E* est *parfaitement compact* si pour toute suite monotone *S* de sous-ensembles *G* de *E* il existe un élément commun à tous les *G* de *S* ou un élément commun aux dérivés de tous les *G* de *S*.

Si un ensemble E est parfaitement compact la suite S, obtenue *en plaçant* E parmi les ensembles G de S est aussi une suite monotone de sous ensembles de E; par suite ou bien il existe un élément *de* E appartenant à tous les G de S ou bien il existe un élément *de* E' qui est élément d'accumulation de tous les G de S. Si, dans le second cas, on peut toujours supposer, quelle que soit la suite S, qu'un élément de E' *et de* E appartient aux ensembles dérivés de tous les G de S, on dira que E est parfaitement compact *en soi.*

Un ensemble parfaitement compact (en soi) est toujours compact (en soi) (XXXII, § 3). La réciproque n'est pas vraie quelle que soit la définition des éléments d'accumulation ; elle n'est pas vraie comme l'avait déja remarqué F. Riesz (XXV) dans l'ensemble des nombres transfinis de la seconde classe, ensemble qui forme une classe (L).

Par contre la réciproque est vraie dans les classes (D), (XXXII, § 5) ; en particulier la condition nécessaire et suffisante pour qu'un ensemble linéaire soit borné est que cet ensemble soit parfaitement compact.

29. *Fonctionnelles.* L' Analyse classique s'occupe essentiellement des fonctions d'une ou de plusieurs variables numériques. Cependant elle présente de nombreux exemples d'expressions qui dépendent d'entités mathématiques plus générales que les points de l'espace linéaire ou de l'espace à une infinité de dimensions. Par exemple l'intégrale $I_f = \int_0^1 f(x)\, dx$ fait correspondre un nombre déterminé I_f à chaque fonction $f(x)$ continue de 0 à 1. De même encore la somme d'une série dépend d'un nombre infini de variables. La conception générale d'une fonction dont la variable pourrait être autre qu'une ou que l'ensemble d'un nombre fini de valeurs numériques s'est faite jour peu à peu au fur et à mesure des besoins de l'Analyse. Les mathématiciens italiens Ascoli, Arzela sont parmi les premiers à avoir étudié les propriétés des fonctions de ligne dont une étude systématique et magistrale a été faite par Volterra. L'étude des fonctions d'une infinité de variables a été entreprise systématiquement d'abord par Le Roux puis par Hilbert, en même temps que la nécessité de cette étude se trouvait manifestée par les travaux de Hill, Poincaré, von Koch, sur les déterminants infinis. De nombreux travaux sont venus s'ajouter à ceux de ces précurseurs et je ne puis songer à en donner la liste. C'est Hadamard qui a attiré le premier l'attention avec le plus de force

(voir par exemple sa communication (IX) au congrès de Heidelberg,) sur l'utilité d'étudier de façon générale ce qu'il appelle une fonctionnelle, c'est à dire une fonction dont la variable, l'argument, est une entité mathématique quelconque. E. H. Moore (XIII) a aussi érigé en corps de doctrine ce qu'il appelle "General Analysis" et qui procéde des mêmes idées.

Nous adopterons aussi la dénomination de fonctionnelle de M. Hadamard. Bien entendu pour pousser logiquement les conséquences de la nouvelle conception, le Calcul Fonctionnel ou ce qu'il vaudrait mieux appeller avec Paul Lévy l'Analyse Fonctionnelle ne devrait pas limiter ses recherches à l'étude des fonctionnelles qui font correspondre un *nombre* à un élément de nature quelconque, mais aussi comme l'a d'ailleurs remarqué Paul Montel, entreprendre l'étude des correspondances générales. C'est toutefois une extension que nous laisseros de coté dans ce mémoire.

SECONDE PARTIE.

CLASSIFICATION ET PROPRIÉTÉS GÉNÉRALES DES ENSEMBLES ABSTRAITS ET DES FONCTIONNELLES.

Pour suivre un ordre logique une théorie des ensembles abstraits devrait débuter par une théorie des nombres cardinaux et ordinaux, par une théorie des groupes abstraits, par une théorie de l'intégrale étendue à un ensemble abstrait (III, IV), théories qui peuvent se développer indépendamment de la notion d'élément d'accumulation. Mais il existe des ouvrages nombreux exposant ces théories ; nous nous contenterons d'y renvoyer le lecteur.

En introduisant la notion d'élément d'accumulation sous son aspect le plus général, puis en assujettissant cette notion à des conditions de plus en plus nombreuses, on est amené à considérer des classes de moins en moins générales mais de plus en plus riches en propriétés et se rapprochant de plus en plus de la classe qui forme l'objet de la Théorie des ensembles linéaires. C'est l'exposition des propriétés de ces classes successives qui va être l'objet de cette Seconde Partie. On aurait pu y joindre la théorie des types de dimensions que nous laisserons aussi de coté (VI, VII, VIII).

CLASSES OÙ LES ÉLÉMENTS D'ACCUMULATION SONT DÉFINIS D'UNE MANIÈRE QUELCONQUE.

31. Supposons que dans une catégorie P d'éléments de nature déterminée, à chaque ensemble E d'éléments de la classe corresponde un ensemble déterminé qu'on appellera *ensemble dérivé* de E et qu'on représentera par $E.'$ (On pourra aussi considérer le cas où à E ne correspond aucun ensemble dérivé et dire dans ce cas indifféremment que E n'a pas d'ensemble dérivé ou que cet ensemble dérivé est nul). On appellera dans ce cas chaque élément de E' un *élément d'accumulation* de E ; l'ensemble dérivé E' de E est l'ensemble des éléments d'accumulation de E.

En employant une notation due à E. H. Moore, *nous appellerons classe d'éléments, le système* (P, K) *qui détermine la catégorie P des éléments que l'on considère et la relation K qui assigne à chaque ensemble E d'éléments de P un certain ensemble dérivé E'* (*qui peut d'ailleurs être nul*).

Théoriquement la relation K est purement arbitraire. Mais il n'y a d'intérêt à considérer des relations K arbitrairement choisies que pour éclaircir cette notion. L' analyse en effet *impose* la considération de certaines relations K convenant chacune à une nature particulière des éléments considérés. C'est seulement pour éviter de nombreuses répétitions d'une part, pour suggérer des méthodes d'investigation et de généralisation, d'autre part, qu'il y a lieu de faire abstraction de la nature des éléments et pour ainsi dire de disséquer les différentes propriétés que peuvent offrir les diverses relations K qui se sont présentées naturellement aux mathématiciens.

32. *Définitions et notations.*—Ajoutons maintenant quelques définitions et notations pour la plupart empruntées à la théorie maintenant classique des ensembles linéaires (I).

L'ensemble des éléments appartenant à un ensemble E ou à un ensemble F s'apelle *somme de E* et de F et se représente par $E+F$.

L'ensemble des éléments d'un ensemble E qui n'appartiennent pas à un ensemble F se représente par $E - F$.

L'ensemble des éléments communs à E et F se représente par $E\times F$ ou $E.\ F$. Lorsqu'il est nul, on dit que *E et F sont disjoints.*

Lorsque tous les éléments de E appartiennent à F, on écrit.

$$E<F \text{ ou } F>E.$$

On appelle *complémentaire* d'un ensemble E, l'ensemble des éléments de la classe considérée qui n'appartiennent pas à E. Plus généralement on pourrait dire que $F - E$ est le complémentaire de E par rapport à un *ensemble fondamental* F.

Une *fonctionnelle uniforme* U_a est définie sur un ensemble E si à tout élément a de E correspond un nombre déterminé U_a appelé valeur de la fonctionnelle U pour l'argument a.

L'oscillation d'une fonctionnelle U_a sur un ensemble I est la différence $\geqq 0$ entre la borne supérieure et la borne inférieure des valeurs prises par U sur I.

Un ensemble E est *fermé* si son dérivé lui appartient

$$E > E'$$

Un ensemble est *dense en soi* s'il appartient à son dérivé

$$E < E'.$$

Un ensemble est *parfait* s'il est à la fois fermé et dense en soi

$$E = E'.$$

Un élément b d'un ensemble E est *isolé* s'il n'appartient pas au dérivé de E. Un ensemble est isolé si chacun de ses éléments est isolé.

Un ensemble E est *séparable* s'il existe un sous-ensemble *dénombrable* N de E tel que tout élément de E appartienne à N ou à son dérivé N' :

$$N < E, \; E < N + N'$$

Un ensemble E est *compact* si tout sous ensemble infini de E a un dérivé non nul. Il est compact *en soi* si tout sous ensemble infini de E donne lieu à au moins un élément d'accumulation appartenant à E.

Une collection d'ensembles est dite *monotone* si de deux ensembles de cette collection (qui est dénombrable ou non), l'un d'eux appartient à l'autre. (En considérant celui ci comme de rang supérieur à celui la, on voit qu'une telle collection est une suite ordonnée).

Ceci étant, un ensemble E est *parfaitement compact* si pour toute suite monotone S de sous ensembles de E, il existe un élément au moins qui est commun à ces sous ensembles ou qui est commun à leurs dérivés. Comme une suite monotone reste monotone si l'on place E en tête, on peut toujours supposer que cet élément appartient à E ou à E'. On dira que E est parfaitement compact *en soi* si l'on peut toujours supposer que cet élément appartient à E.

Un élément b est un *élément de condensation* d'un ensemble E s'il est élément d'accumulation non seulement de E mais de tous les ensembles $E-N$ obtenus en supprimant de E un ensemble dénombrable N quelconque d'éléments. D'après cela, seul un ensemble non dénombrable peut avoir un élément de condensation.

Un ensemble E est dit *condensé* si tout sous ensemble non dénombrable d'éléments de E donne lieu à un élément de condensation.

Un élément b est dit *intérieur* à un ensemble E si non seulement il appartient à E mais encore s'il n'est élément d'accumulation d'aucun sous-ensemble du complémentaire de E. Un élément est *extérieur* à E s'il est intérieur à l'ensemble complémentaire de E. La *frontière* de E est constituée par l'ensemble des éléments qui ne sont ni intérieurs n extérieurs à E.

Un ensemble est *ouvert* si chacun de ses éléments lui est intérieur.

Un ensemble E possède la propriété de Lindelöf si F étant unt famille quelconque d'ensembles I tel que tout élément de E soie intérieur à l'un des I, la même propriété appartient à une certaine famille F_1 *dénombrable* extraite de F.

Si l'on peut quel que soit F, supposer F_1 fini, on dit que E posséde la propriété de Borel-Lebesgue. Si l'on peut supposer F_1 fini, quand F est une famille dénombrable, on dit que E posséde la propriété de Borel.

On voit que la propriété de Borel-Lebesgue est équivaléente à l'ensemble des propriétés de Lindelöf et de Borel.

Deux ensembles E, F sont *enchaînés* l'un à l'autre si un élément de l'un est élément d'accumulation de l'autre ou s'ils ont un élément d'accumulation commun ; autrement dit si

$$E'.\ F+E\ .\ F'+E'.\ F'\neq 0$$

Un ensemble est *bien enchaîné* si lorsqu'on le considére de toutes les manières possibles comme la somme de deux ensembles distincts non nuls, ces deux ensembles (disjoints ou non) sont toujours enchaînés l'un à l'autre.

Un *continu* est un ensemble fermé, bien enchaîné et auquel appartient plus d'un élément.

Le *composant* d'un ensemble E relatif à un élément b de E est la somme de tous les sous-ensembles de E qui contiennent b et sont bien enchaînés. (Il peut arriver que ce composant se réduise à b.)

CLASSES QUELCONQUES.

33. Quelle que soit la relation EKE' (§ 31) qui définit les ensembles dérivés :

Tout ensemble parfaitement compact (en soi) est compact (en soi).

Tout partie d'un ensemble (parfaitement) compact est un ensemble (parfaitement) compact.

Si un ensemble contient un ensemble (parfaitement) compact, il n'est pas lui même (parfaitement) compact.

Un ensemble (parfaitement) compact qui est fermé est (parfaitement) compact en soi.

34. *Fonctionnelles continues dans des classes quelconques.*—Etant donnée une fonctionnelle V uniforme sur un ensemble E parfaitement compact en soi, il existe au moins un élément a_0 de E tel que la borne supérieure, finie ou non de V sur E soit égale à la borne supérieure de V sur la partie de E appartenant à un ensemble quelconque auquel a_0 est intérieur (XXXII, § 19).

Cet énoncé suggère la définition suivante des fonctionnelles *continues.*

Une fonctionnelle U est continue sur E en un élément a_0 de E si la borne inférieure de l'oscillation de U sur la partie de E qui appartient à un ensemble I est nulle quand on fait varier I de sorte que a_0 lui reste intérieur.

Il est en effet équivalent de dire que : Ua_0 doit être toujours égal à l'une des valeurs ou l'une des limites des valeurs prises par U sur un sous ensemble quelconque de E ayant a_0 pour élément d'accumulation (XXXII § 17).

Moyennant la première définition, on voit que : si une fonctionnelle U est continue partout sur un ensemble E parfaitement compact en soi, 1° cette fonctionnelle est bornée sur E, 2° elle atteint en au moins un élément de E sa borne supérieure et en au moins un élément de E sa borne inférieure.

Plus généralement, on peut appeler fonctionnelle *semi continue supérieurement* sur E en a_0, une fonctionnelle dont la valeur Ua_0 en a_0 est égale à la borne inférieure, quand l'ensemble I varie de sorte que a_0 lui reste intérieur, de la borne supérieure des valeurs prises par U sur la partie de E qui appartient à I. Il est équivalent de convenir que Ua_0 doit être au moins égal à l'une des valeurs ou l'une des limites des valeurs, prises par U sur un sous-ensemble quelconque de E admettant a_0 pour élément d'accumulation.

Ceci étant : Toute fonctionnelle semi-continue supérieurement partout sur un ensemble E parfaitement compact en soi est bornée supérieurement sur E et atteint sa borne supérieure en au moins un élément de E.

35. Les fonctionnelles d'une famille sont dites *également continues* en a_0 sur E si quel que soit le nombre posilif ϵ, il existe un ensemble I_0 auquel a_0 est intérieur, tel que l'oscillation de chaque fonctionnelle de F sur la partie de E qui appartient à I_0 soit $\angle \epsilon$. Bien entendu il en rèsulte que chacune des fonctionnelles de F est continue en a_0 sur E, mais la réciproque n'est pas vraie.

L'intérêt de la notion d'égale continuité réside dans le théorème suivant :

Etant donnée une famille F de fonctionnelles bornées dans leur ensemble et également continues en tout élément d'un ensemble séparable E, sur E. il existe une suite de fonctionnelles extraite de F qui converge sur E vers une fonctionnelle continue en chaque élément de E sur E. Et la convergence est uniforme sur tout sous-ensemble de E qui est parfaitement compact en soi.

Pour démontrer ce théoréme, il est bon de commencer par démontrer les lemmes suivants qui ont aussi leur intérêt (XXXII, § 20).

Considérons une suite de fonctionnelles $U^{(1}$, $U^{2)}$....qui converge sur un ensemble E :

Si ces fonctionnelles sont également continues en a_0 sur E, leur limite est continue en a_0 sur E.

Si ces fonctionnelles sont également continues partout sur E et si E est parfaitement compact en soi, la convergence est nécessairement uniforme sur E.

Si ces fonctionnelles sont également continues partout sur l'ensemble $E+E'$, quel que soit l'ensemble E, elles convergent aussi sur $E+E$

36. Etant donnée une famille F de fonctionnelles U définies sur un ensemble E, appelons T_b la borne supérieure des valeurs des fonctionnelles U de F en b Si ces fonctionnelles sont bornées dans leur ensemble en chaque élément b de E, la fonctionnelle T sera bien définie sur E . si les fonctionnelles U de F sont également continues en a_0 sur

E, leur borne supérieure T est une fonctionnelle continue en a_0 sur E (XXXII, § 20 bis).

CLASSES (V).

37. Nous appellerons *classes* (V) une classe d'éléments où la relation EKE' qui détermine les ensembles dérivés est définie de la manière suivante (X, page 3).

A tout élément a on fait correspondre une certaine famille $\{V_a\}$ d'ensembles V_a qu'on appelle *voisinages* de a. L'élément a sera considéré comme élément d'accumulation d'un ensemble E si celui-ci a des éléments aussi voisins que l'on veut de a, c'est à dire s'il a un élément au moins, *distinct de* a, en commun avec chaque voisinage V_a de a.*

On peut ajouter ou retrancher l'élément a à chacun de ses voisinages sans altérer la relation EKE'. Bien que ce ne soit pas indispensable, mais pour nous rapprocher du cas des ensembles linéaires nous conviendrons dans la suite que chaque élément appartient à chacun de ses voisinages.

On peut encore définir les classes (V) par un procédé descriptif (X, page 3) en appelant classe (V) une classe d'éléments où la relation EKE' satisfait aux conditions :

1° $(E+F)' > E'+F'$

(b) un élément d'accumulation a d'un ensemble E est aussi élément d'accumulation de l'ensemble E a.

Si ces deux conditions sont remplies, on peut obtenir la même relation K en définissant les éléments d'accumulation au moyen de voisinages convenablement choisis, pour lesquels on pourra prendre par exemple comme voisinages de a tous les ensembles auxquels a est intérieur (X, page 4).

Quand on utilise la conception de voisinage, on peut énoncer ainsi certaines définitions relatives aux fonctionnelles.

Une fonctionnelle U est continue sur E en un élément a_0 de E si son oscillation sur la partie de E qui appartient à un voisinage de a^0 a une borne inférieure nulle quand on fait varier ce voisinage de a_0.

Les fonctionnelles formant un certaine famille F sont dites également continues en a_0 sur E, s'il existe quel que soit $\epsilon > 0$ un voisinage V^0 de a_0 tel que l'oscillation de chaque fonctionnelle de F sur la partie de E qui appartient à V^0 soit $< E$.

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* La notation classe (V) et l'expression voisinage sont donc employés dans un sens différent de celui que j'avais adopté dans ma Thèse (XVIII, page 18) et que j'ai abandonné en 1918 (X, page 3).

38. *Propriétés des classes* (V).—La somme d'un nombre fini d'ensembles (parfaitement) compacts est un ensemble (parfaitement) compact.

L'ensemble commun à des ensembles fermés est fermé. La somme d'ensembles denses en soi est dense en soi. Tout ensemble possédant la propriété (§ 32)

de Lindelöf est condensé (XXXII, § 14).

de Borel est compact en soi (X, § XIX, page 15)

de Borel-Lebesgue est parfaitement compact en soi (XXXII, § 8).

L'ensemble des éléments isolés d'un ensemble séparable (§ 32) est dénombrable. En particulier tout ensemble isolé est dénombrable.

Par définition un ensemble séparable E appartient à la somme $N+N'$ d'un des sous-ensembles dénombrables N de E et de son dérivé N' : quand E est fermé $E=N+N'$.

CLASSES (H).

39. Une classe (H) est une classe (V) particulière ; c'est une classe où les éléments d'accumulation sont définis par le moyen de familles de voisinage $\{V_a\}$ satisfaisant aux conditions suivantes (a étant supposé appartenir à V_a) (XXXII, § 23).

2° bis. Si $V_b{}^1$, $V_b{}^2$ sont deux voisinages quelconques de b il existe un voisinage de b qui appartient entièrement à la fois à $V_b{}^1$ et à $V_b{}^2$.

5° ter. Si c est un élément quelconque de l'un V_b des voisinages de b, il existe un voisinage de c qui appartient à V_b.

3° bis. Étant donnés deux éléments distincts b, c, il existe au moins un voisinage de b auquel c n'appartient pas et inversement.

Il faut d'ailleurs remarquer que si l'on remplace les familles de voisinages des différents éléments par des familles équivalentes, 2° bis et 3° bis subsisteront : mais 5° ter n'est vérifié que par un choix convenable des familles équivalentes sans lequel, seule subsiste, au lieu de 5° ter, la condition

5° bis Si V_b est un voisinage quelconque de b, il existe un voisinage $V_b{}'$ de b tel que pour tout élément c de $V_b{}'$, il existe un voisinage de c appartenant à V_b.

On peut encore donner des classes (H) une définition descriptive en les définissant comme des classes d'éléments satisfaisant aux conditions 1°, 2°, 3° de F. Riesz et à la condition 5° mentionnées au § 18. Nous allons donc ranger ces conditions parmi les propriétés des classes *H*, mais nous nous souviendrons de plus que cet ensemble de quatre propriétés *caractérise* les classes (H).

40. *Propriétés des classes* (*H*).--

1° et 2°. La dérivation des ensembles est une opération distributive ; en d'autres termes

$$(E+F)' = E'+F'.$$

3°. Un ensemble ne comprenant qu'un nombre fini d'éléments n'a aucun élément d'accumulation.

5° Tout ensemble dérivé est fermé.

Remarque. L'ensemble de ces trois propriétés caractérise les classes (*H*).

Soit E_1, E_2,....E_n....une suite d'ensembles contenant chacun le suivant : si le premier est compact (et par suite aussi les suivants), il y a un élément commun à tous les E_n, ou un élément commun aux dérivés des E_n (XXXII, § 5). En particulier si E_1, E_2,.....sont chacun compact en soi, ils ont certainement un élément commun.

La somme d'un nombre fini d'ensembles fermés est fermée.

Quel que soit l'ensemble E, parmi tous les ensembles fermés qui contiennent E, il en existe un qui appartient à tous ceux-ci, c'est l'ensemble $E+E'$. On peut l'appeler (avec F. Hausdorff, qui a établi cette proposition et la suivante dans un cas moins général), le plus petit des ensembles fermés contenant E (XXXII, § 25).

Quel que soit l'ensemble E s'il existe un élément commun à E et à E', il existe au moins un ensemble dense en soi et appartenant à E ; la somme de tous les ensembles de ce genre est aussi dense en soi et c'est l'ensemble commun à E et à E'. On peut aussi l'appeler avec F. Hausdorff, le plus grand des sous ensembles denses en soi de E (XXXII, § 25).

L'ensemble des éléments non intérieurs à un certain ensemble est fermé.

Tout ensemble séparable dense en soi appartient au dérivé d'un de ses sous-ensembles dénombrables.

Tout ensemble séparable parfait est le dérivé d'un de ses sous ensembles dénombrables.

Si un élément b est intérieur à un ensemble E et s'il appartient au dérivé d'un ensemble F, il existe un sous ensemble F de E dont tous les éléments sont intérieurs à E et dont b est élément d'accumulation. Cette propriété reste vraie si on suppose simplement sur E que b n'est pas élément d'accumulation de l'ensemble complémentaire de E. Inversement si cette dernière propriété est vérifiée par une classe (V), celle-ci vérifie nécessairement les conditions 1°, 2° et 5° (X, page 17).

On se sert de cette propriété pour démontrer que : La condition nécessaire et suffisante pour qu'un ensemble possède la propriété de Borel est qu'il soit compact en soi (X, page 19).

La condition nécessaire et suffisante pour qu'un ensemble possède la propriété de Borel-Lebesgue est qu'il soit parfaitement compact en soi (XXXII, § 9).

[Par analogie, l'énoncé suivant, vrai dans une classe (V) en ce qui concerne la condition nécessaire et au moins dans une classe (D) (§ 46), en ce qui concerne la condition suffisante, ne serait-il pas vrai dans une classe (H) ?

La condition nécessaire et suffisante (?) pour qu'un ensemble possède la propriété de Lindelöf est que cet ensemble soit condensé]

La condition nécessaire et suffisante pour qu'un ensemble soit enchainé à la somme $E+F$ de deux ensembles est qu'il soit enchainé à l'un ou à l'autre.

Deux ensembles composés chacun d'un nombre fini d'éléments ne peuvent être enchaînés.

Pour qu'un élément soit enchaîné à un ensemble il faut et il suffit qu'il en soit élément d'accumulation.

Deux ensembles enchaînés à un même élément sont enchaînés.

La condition nécessaire et suffisante pour qu'un ensemble E fermé soit bien enchainé est qu'il ne puisse être décomposéen deux ensembles fermés disjoints. Si E n'est pas fermé, cette condition doit être remplie par $E+E'$ qui est fermé. Ceci peut servir (dans une classe (H)) de définition des ensembles bien enchaînés (XXXII, § 28).

La condition nécessaire et suffisante pour qu'un ensemble E soit bien enchaîné est que deux quelconques de ses éléments appartiennent à un ensemble bien enchaîné qui soit sous ensemble de E (XXXII, § 28).

La somme de deux ensembles chacun bien enchaînés qui ont un élément commun ou qui sont enchaînés l'un à l'autre est bien enchaînée. Plus généralement :

La somme d'ensembles bien enchaînés qui deux à deux ont un élément commun ou sont enchaînés est bien enchaînée (XXXII, § 28). Tout ensemble est la somme d'ensembles bien enchaînés (ou réduits à un seul élément), disjoints et dont deux quelconques ne sont jamais enchaînés, à savoir, ses composants. Réciproquement si un ensemble E est la somme d'un *nombre fini* d'ensemble bien enchaînés (ou réduits à un seul élément) disjoints et dont deux quelconques ne sont jamais enchaînés, ces sous ensembles sont les composants de E (XXII, § 29).

Si un continu a un élément au moins en commun avec un ensemble G sans lui appartenir entièrement il a aussi un élément au moins un commun avec la frontière de G (XXXII, § 29).

Si un ensemble E est bien enchaîné et s' il contient plus d'un élément, il est dense en soi ; il est donc contenu dans son dérivé E'. En ajoutant à un ensemble bien enchaîné E des éléments d'accumulation de E, celui-ci reste bien enchaîné. En particulier, E' est bien enchaîné ; il est d'ailleurs fermé ; donc : tout ensemble bien enchaîné fait partie d'un continu, par exemple son propre dérivé.

Tout continu est un ensemble parfait.

Tout composant d'un ensemble fermé est un continu ou se réduit à un élément.

Appelons arc de Jordan d'extrémités b, c un ensemble qui correspond de façon biunivoque et bicontinue avec un segment de droite fermé et limité dont les extrémités correspondent à b et c. (Une correspondance entre deux ensembles E, F est bicontinue si à tout élément d'accumulation b d'un sous-ensemble E_1 de E correspond un élément d'accumulation c du sous ensemble F_1 de F qui correspond à E_1.

Si pour tout couple d'éléments d'un ensemble E, il existe un arc de Jordan appartenant à E et ayant ces deux éléments pour extrémités, E est bien enchaîné.

41. *Limite de fonctionnelles continues.*—Si une suite de fonctionnelles. $U^{(n)}$ converge sur un ensemble E sur lequel elles sont partout continues et si l'ensemble E est parfaitement compact en soi, il faut pour que la limite U des fonctionnelles $U^{(n)}$ soit partout continue sur E que la convergence soit quasi-uniforme sur E ;

quel que soit l'ensemble E, cela suffit (XXXII, § 22).

(La convergence sera dite quasi-uniforme sur E si quels que soient les nombres positifs ϵ et N, un nombre $N' \geqq N$ existe tel que l'on ait quel que soit l'élément b de E

$$| U_b - U_b^{(n)} | \angle \epsilon$$

pour une valeur au moins de n, variant peut être avec b mais comprise entre N et N' : $N \underline{\angle} n \underline{\angle} N$).

41 bis. *Valeurs intermédiaires.*—Une fonctionnelle continue partousur un ensemble continu ne peut y prendre deux valeurs distinctes sans passer sur cet ensemble aussi près que l'on veut des valeurs intermédit aires.

Une fonctionnelle continue partout sur un ensemble continu et compact ne peut y prendre deux valeurs sans prendre également toutes les valeurs intermédiaires (XXXII, § 29).

Classes (L)

42. Arrivons maintenant au cas d'une loi de correspondance entre ensembles et ensembles dérivés se rapprochant plus du cas linéaire que celui des H classes ou celui des classes de Hausdorff.

C'est le cas que j'avais considéré dans ma Thèse:

Supposons une loi qui permette de distinguer parmi les suites infinies $a_1, a_2, \dots$ d'élément distincts ou non, celles dites convergentes et de leur assigner un élément dit élément-limite de la suite. Supposons que cette loi satisfasse aux deux conditions suivantes :

1° une suite composée d'éléments identiques à a est convergente et a a pour élément-limite.

2° une suite extraite d'une suite convergente est convergente et a le même élément limite.

On pourra alors définir élément d'accumulation d'un ensemble E tout élément qui soit élément-limite d'une suite convergente extraite de E. Si on ne peut extraire de E aucune suite convergente, E' esr nul.

On voit alors facilement que les conditions 1°, 2°, 3°, 4° de F. Riesz (§ 17) sont satisfaites. La condition 5° peut ou non être satisfaite par une classe (L), comme le montre l'exemple suivant.

Considérons comme éléments les fonctions d'une variable réelle $f(x)$ dans l'intervalle fini I. Disons que $f_1, f_2, \dots$ forme une suite convergente et a $f(x)$ pour élément limite, s'il existe une fonction $f(x)$ telle que $f_n(x) \longrightarrow f(x)$ pour chaque valeur de x dans I. On voit facilement que cette classe est (L). Elle ne vérifie pas 5° car si l'on prend pour E l'ensemble des fonctions continues on sait que E' ne sera pas fermé.

43. *Propriétés des classes* (L).—Une fonctionnelle U est continue en a_0 sur E si quelle que soit la suite convergente $a_1, a_2, \ldots a_n \ldots$ d'éléments de E tendant vers a_0, les valeurs de U sur cette suite convergent vers sa valeur $U a_0$ en a_0 (X, page 14).

Des fonctionnelles U sont également continues en a_0 sur E, si pour toute suite convergente $a_1, a_2, \ldots$ d'éléments de E tendant vers a_0 les valeurs de U sur cette suite diffèrent d'aussi peu que l'on veut de sa valeur $U a_0$ en a_0, à partir d'un certain rang indépendant de celle des fonctionnelles que l'on considère (XVIII, page 11).

Soit F une famille de fonctionnelles continues partout sur un ensemble E compact et fermé ; pour que de toute famille infinie F_1 extraite de F, on puisse extraire une suite de fonctionnelles qui convergent uniformément sur E (vers une fonctionnelle nécessairement continue partout sur E) ; il faut et il suffit que les fonctionnelles de F soient bornées dans leur ensemble et également continues partout sur E. Il a été démontré que la condition était suffisante sans rien supposer sur la définition des éléments d'accumulation (XXXII,) mais en supposant E parfaitement compact en soi, ce qui dans le cas actuel d'une classe (L) est équivalent à : E compact et fermé. On peut aussi démontrer sans difficulté et sous la même hypothèse générale qu'il est nécessaire que les fonctionnelles soient bornées dans leur ensemble. Le reste de la condition nécessaire a été obtenu dans ma Thèse (XVIII, § 18, page 13) dans le cas plus étroit de l'énoncé (classe (L)) mais il parait certain qu'on doit pouvoir étendre la démonstration, en la modifiant, au cas d'une classe (H) tout au moins.

Puisqu'une classe (L) satisfait aux conditions 1°, 2°, 3°, 4°, c'est aussi une classe (V) et on peut y définir les éléments d'accumulation au moyen de voisinages convenablement choisis (par exemple on peut prendre comme voisinages de l'élément a_0 les ensembles auxquels a_0 est intérieur). Relativement à ces voisinages :

vers l'éléments limite a_0, sont tous contenus à partir d'un certain rang (variable) dans chaque voisinage de a_0.

Les voisinages de chaque élément forment une suite toujours dénombrable, pour certaines classe (L).

La propriété n'est pas toujours vraie : même en remplaçant au besoin les familles des voisinages de chaque élément d'une classe (L) par des familles équivalentes, on ne peut toujours supposer dénombrable chacune de ces familles : tel est le cas de la classe (L) des fonctions réelles (XI, page 59).

Les conditions nécessaires et suffisantes pour qu'il en soit ainsi sont les suivantes : (α) toute suite convergente reste convergente quand on lui adjoint un nombre fini d'éléments ; (β) pour tout élément non isolé a_0, toute suite σ d éléments a_1^{σ}, a_2^{σ}, ... a_n^{σ} .... et tout entier n, il existe un entier Q_n^{σ} tel que : σ_1, σ_2, ... étant une suite de suites (distinctes ou non, convergeant vers a_0 et n_1, n_2, ... une suite d'entiers, la suite $a_{n_1}^{\sigma_1}$, $a_{n_2}^{\sigma_2}$, .. converge vers a_0, si quelque soit N, i_0 existe de sorte que pour $i > i_0$, on ait $n_i > Q_N^{\sigma_i}$. Ces deux conditions (α), (β) sont d'ailleurs indépendantes (XI, page 58).

Un ensemble (parfaitement) compact en soi est nécessairement fermé. De sorte que dans les classes (L), il y a identité entre les notions d'ensemble (parfaitement) compact en soi d'une part et d'ensemble (parfaitement) compact et fermé, d'autre part.

Tout ensemble séparable a au plus la puissance du continu (XXXII, § 14).

CLASSES (S).

44. Nous appellerons classe (S) une classe (L) vérifiant la condition.

5° Tout ensemble dérivé est fermé.

Il est entièrement *équivalent* (XVI, page 3) d'appeler classe (S) une classe (L) dans laquelle tout ensemble compact et fermé possède la propriété de Borel.

Si b est intérieur à un ensemble I et si b est élément limite d'une suite convergente d'éléments a_1, a_2, .. , les éléments de cette suite sont intérieurs à I à partir d'un certain rang.

Il en est de même si b sans appartenir à I est intérieur à $I+b$ (XXI).

CLASSES (E).

45. Une classe (E) est une classe où la relation EKE' qui détermine les ensembles dérivés est définie par l'intermédiaire de la notion d'écart.

A tout couple b, c d'éléments de la classe correspond par hypothèse un nombre (b, c) appelé écart de b et de c et qui satisfait aux conditions suivantes

(I) $(b, c) = (c, b) \geqq 0$,

(II) l'égalité $(b \; c) = 0$ se présente lorsque b et c ne sont pas considérés comme distincts,

(III) un élément b est élément d'accumulation d'un ensemble E s'il existe une infinité d'éléments de E, qui sont distincts et dont les distances à b tendent vers zéro.

En appelant sphéroïde de centre a_0 et de rayon ρ, l'ensemble des éléments b tels que $(a_0, b) \leqq \rho$, on voit qu'on peut considérer une classe (E) comme une classe (V) dans laquelle à chaque élément a_0 est attachée une famille *dénombrable* de voisinages, à savoir les sphéroïdes de centre a_0, rayons $1, \frac{1}{2}, \frac{1}{3} \dots \frac{1}{p} \dots$.

Réciproquement considérons une classe (V) où la famille de voisinages attachée à chaque élément est équivalente à une famille dénombrable. Pour qu'une telle classe soit une classe (E), il faut et il suffit qu'il existe pour chaque élément b une famille dénombrable de voisinages $T_b^1, T_b^2, \dots T_b^p, \dots$ équivalente à la famille donnée attachée à b et telle que pour tout entier N et tout élément b on puisse déterminer un entier m pour lequel le voisinage T_c^N contient nécessairement b, si c appartient à T_b^m (XI, § 14).

D'ailleurs la condition que nous venons d'énoncer est une véritable condition ; elle n'est pas satisfaite d'elle même : On peut citer un exemple de classe (V) où les familles de voisinages attachées à chaque élément sont dénombrables, exemple choisi même de façon que cette classe soit une classe (L), et qui pourtant n'est pas une classe (E). C'est l'exemple de la classe des points d'une droite où l'on n'admet comme suites convergentes que celles qui convergent à la manière ordinaire mais, à partir d'un certain rang de la suite, jamais à gauche du point limite. On pourra alors choisir comme voisinages d'un élément x_0, les intervalles de longueur $1, \frac{1}{2}, \dots \frac{1}{p}, \dots$ ayant leurs extrémités droites confondues avec x_0 (XI, § 12).

Remarquons aussi que si une classe (E) est une classe (L) ce n'est pas nécessairement une classe (S), autrement dit un ensemble dérivé n'y est pas nécessairement fermé (XI; § 1).

Inversement une classe (S) n'est pas nécessairement une classe (E) (XI, § 6).

Pour qu'une classe (E) soit une classe (S), autrement dit pour que tout ensemble dérivé soit fermé dans une classe où les ensembles dérivés sont définis par le moyen d'un écart, il faut et il suffit qu'à tout élément b et tout $\delta > 0$ corresponde η tel que c étant un élément quelconque pour lequel $(b, c) < \eta$ et ω étant un certain nombre déterminé par δ, η, b, c, on ait

$$(b, d) < \delta \text{ pour } (c, d) < \omega.$$

Classes (D).

46. On appelle classe (D) une classe d'éléments où la relation EKE' qui définit les ensembles dérivés est définie de la façon suivante :

A tout couple d'éléments b, c correspond un nombre représenté par la notation (b, c), appelé distance de b et de c et satisfaisant aux conditions suivantes :

(I) $(b, c) = (c, b) \geqq 0$

(II) il est équivalent d'écrire $(b, c) = 0$ ou de dire que b et c ne sont pas considérés comme distincts.

(III) la condition nécessaire et suffisante pour qu'un élément b soit élément d'accumulation d'un ensemble E est que la limite inférieure des distances de b aux éléments de E soit nulle.

(IV) quels que soient les éléments a, b, c

$$(a, b) \leqq (a, c) + (c, b).$$

Les classes (E), plus générales que les classes (D) sont celles qu'on obtient quand on n'impose pas la condition IV.

Une classe (D) est donc une classe (E) ; c'est aussi une classe (L) comme on le voit en appelant suite convergente vers a_0, une suite $a_1\ a_2, \ldots$ dont les distances à a_0 tendant vers zéro.

C'est aussi une classe (V) comme on le voit en appelant famille de voisinages attachés à un élément a_0 la famille des sphéroïdes de centre a_0 et de rayons $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots$ (On appelle sphéroïde de centre a_0, rayon $\rho > 0$, l'ensemble des éléments b tels que $(a_0\ b) \leqq \rho$). Comme les conditions $2^{\circ\text{ bis}}$, $3^{\circ\text{ bis}}$, $4^{\circ\text{ bis}}$ $5^{\circ\text{ bis}}$ sont évidemment satisfaites par ces voisinages, on voit qu'une classe (D) est aussi une classe (H) et une classe (S). C'est donc un cas particulier de toutes les classes qui viennnent d'être envisagées dans cette Seconde Partie. C'est aussi un cas particulier de l'espace topologique de Hausdorff (§ 26). Malgré une apparente plus grande généralité les classes que j'avais appelé classes (V) dans ma Thèse (§ 11^{bis}), sont d'après Chittenden des classes (D) (XXIII). C'est pourquoi j'emploie maintenant l'expression de classe (V) et de voisinage dans un sens différent.

47. *Propriétés des classes* (*D*) —Dans une telle classe:

Il y a identité entre les notions d'ensemble parfaitement compact (en soi) et d'ensemble compact (en soi), (XXXII, § 6)

Il y a aussi identité dans ce cas entre les notions d'ensemble séparable et d'ensemble condensé, (XXXII, § 14).

Etant donné un ensemble E, il est toujours possible et d'une infinité de manières d'imaginer pour tout nombre positif ϵ, une famille K_ϵ d'ensembles k_ϵ tel que. 1° la distance de deux éléments d'un de ces k_ϵ reste toujours $< \epsilon$. 2° tout élément de E appartient à l'un de ces k_ϵ. On peut dire avec M. Hadamard (IX) que la puissance de K_ϵ "numère" E. Les deux résultats suivants ont donc leur intérêt à ce point de vue :

La condition nécessaire et suffisante pour que quel que soit $\epsilon > 0$, l'une au moins des familles K_ϵ soit dénombrable est que E soit séparable (XXXII, § 14).

La condition nécessaire et suffisante pour que quel que soit $\epsilon > 0$, l'une au moins des familles K_ϵ soit finie est que E soit compact. (XXXII, § 14).

Les résultats précédents restent exacts si l'on remplace dans la deuxième condition imposée aux k_ϵ le mot "appartient" par "est intérieur." Ils restent aussi exacts si l'on assujettit les k_ϵ à être des sphéroïdes ayant pour centres des éléments de E.

Ajoutons aussi les propriétés suivantes (XXXII, § 14).

Toute partie d'un ensemble séparable est un ensemble séparable.

L'ensemble dérivé d'un ensemble séparable est séparable

Tout sous ensemble fermé et séparable d'un ensemble G dense en soi peut être considéré comme le dérivé d'un sous ensemble dénombrable de G.

Un ensemble compact est séparable (la réciproque n'est pas vraie).

Et même (XVII, § 32), la somme d'une infinité dénombrable d'ensembles compacts est séparable (la réciproque n'est pas vraie comme on le voit en prenant pour ensemble séparable l'espace (Ω) du § 60),

Tout sous ensemble fermé d'un ensemble séparable E peut être obtenu en supprimant de E les éléments appartenant à un certain ensemble dénombrable de sphéroïdes.

La condition nécessaire et suffisante pour qu'un ensemble possède la propriété de Lindelöf est que cet ensemble soit séparable.

Tout ensemble fermé et séparable F est la somme d'un ensemble dénombrable N et d'un ensemble parfait P. L'ensemble parfait P peut être caractérisé comme l'ensemble des éléments de condensation de F, ou comme l'ensemble commun à tous les ensembles dérivés de F d'ordre fini ou transfini. Il y a d'ailleurs un rang fini ou transfini, α, à partir duquel tous les dérivés de F sont identiques à P. L'ensemble dénombrable N peut être représenté par la somme finie ou transfinie

$$N = (F - F^1) + \ldots\ldots = \sum_{n=0}^{n-a}\left[F^{(n)} - F^{(n+1)}\right]$$

Et aussi (XIX, § 6) :

L'ensemble dérivé d'un ensemble compact est compact. Et par conséquent :

Tout ensemble compact E fait partie d'un ensemble compact et fermé, à savoir $E + E.'$

Pour qu'une fonctionnelle U soit continue en a_0 sur l'ensemble E, il faut et il suffit qu'à tout nombre $\epsilon > 0$, corresponde un nombre η tel que pour tout élément b de E, l'inégalité

$$(a_0, b) < \eta \text{ entraîne } / U_{a_0} - U_b / < \epsilon.$$

Pour que des fonctionnelles soient également continues en a_0 sur l'ensemble E, il faut et il suffit que le nombre η qu'on vient de définir puisse être choisi indépendamment de celle des fonctionnelles considérées qui figure dans la seconde inégalité.

Une fonctionnelle est *dite uniformément continue* sur un ensemble E lorsque le nombre η défini plus haut peut être choisi indépendamment de l'élément a_0.

C'est la notion de distance (ou d'écart) qui permet de définir la notion de *continuité uniforme* dont le sens ne parait pas très aisé à établir autrement Il faut bien remarquer en effet qu'une fonctionnelle continue sur un ensemble peut y être uniformément continue ou non suivant qu'on y définit la distance d'une façon ou d'une autre, même avec des définitions qui n'altèrent pas, quand on passe de l'une à l autre la relation E K E' entre ensembles et ensembles dérivés.

Par exemple, les éléments d'accumulation des ensembles linéaires ne sont pas changés quand on remplace la distance ordinaire de deux points par la distance des deux points correspondants dans une inversion déterminée. Or la fonction qui représente l'abscisse est uniformément continue dans un cas, continue non uniformément dans l'autre.

Toute fonctionnelle continue sur un ensemble compact et fermé est uniformément continue sur cet ensemble (XVIII, page 29, § 47).

Il en résulte alors que pour que des fonctionnelles U formant une famille F soient également continues partout sur un ensemble compact et fermé E, il faut et il suffit que le nombre η défini plus haut puisse être choisi indépendamment de la fonctionnelle U dans F et indépen damment de a_0 sur E (XVIII, page 29, § 48).

Etant donnés (XVIII) un ensemble quelconque E, il existe au moins une fonctionnelle U partout continue sur E sãns y être constante (si E a plus d'un élément. Il serait interessant de déterminer des classes ces plus générales jouissant de cette propriété.

La condition nécessaire et suffisante pour que toute fonctionnelle continue sur un ensemble E y soit bornée et y atteigne sa borne supérieure et sa borne inférieure est que E soit compact et fermé (XVIII, page 31, § 51). Il serait intéressant d'étendre ce théorème à des classes plus générales que les classes (D), si cela est possible ou inversement de chercher quelles sont les classes où il est vrai.

Si deux ensembles sont enchaînés, la limite inférieure de la distance d'un élément de l'un à un élément de l'autre est nulle. La réciproque est vraie si l'un au moins des deux ensembles est compact (XXXII, § 24 bis). Il faut remarquer que nous distinguons ici la limite inférieure de la borne inférieure.

Quel que soit $\epsilon > 0$, on peut joindre deux éléments quelconques d'un ensemble bien enchaîné par une chaîne à maillons $< \epsilon$, c'est à dire extraire de l'ensemble une suite ordonnée d'un nombre fini d élémentx dont les deux extrèmes sont les éléments donnés et dont deus consécutifs sont à une distance $< \epsilon$. Réciproquement si deux éléments quelconques d'un ensemble peuvent être, quel que soit ϵ, joints par une chaîne à maillons $< \epsilon$, et si l'ensemble est compact, il est bien enchaîné.

48. En particulier considérons celles des classes (D) *dans lesquelles* tout couple d'éléments appartenant á un même sphéroïde peuvent être joints par un arc de Jordan appartenant à ce sphéroïde et ayant pour extrémités ces éléments. On verra plus loin (§ 52–62) que les classes

les plus importantes considérées en Analyse satisfont à cette condition.

Sous cette hypothése :

La classe considérée est parfaite et forme un continu.

Un ensemble ne peut être à la fois ouvert et fermé.

Les composants d'un ensemble ouvert sont des ensembles ouverts.

Un ensemble bien enchaîné reste bien enchaîné quand on lui ajoute des composants de l'ensemble complémentaire.

49. *Classe* (*D*) *parfaite.*—Dans une classe (D) parfaite, il y a identité entre les ensembles fermés et les ensembles dérivés. Et même, non seulement tout ensemble dérivé est fermé, mais tout ensemble fermé peut être considéré comme le dérivé d'un ensemble dénombrable (XXXII, § 14),

50. *Classe* (*D*) *compléte.*—Dans une classe (D) compléte (§ 10) :

un ensemble parfait quelconque n'est jamais dénombrable (XIX, § 14, page 18).

un ensemble parfait séparable a exactement la puissance du continu (XVIII, page 808).

Pour qu'un ensemble E soit compact, il faut et il suffit que (la définition de la distance étant convenablement choisie), tout sous ensemble E_k de E où la distance reste à $k > 0$ soit composé d'un nombre fini d'éléments pour toute valeur de k. La condition est d'ailleurs nécessaire même si la classe n'est pas compléte (XIX, § 4).

50 bis Dans un mémoire actuellement sous presse (XXXVIII, § 5) N. Wiener introduit une conception intéressante, celle des classes (D) *vectorielles*, c'est à dire des classes (D) où à tout couple d'éléments a, b correspond une entité a b jouissant des propriétés de composition des vecteurs et où la *distance* (a, b) jouit des propriétés de la longueur du vecteur a b. Il introduit aussi la conception de classes (D) qui ne sont vectorielles que localement.

Il serait intéressant de chercher à quelle condition une classe (D) vectorielle séparable est homéomorphe de la classe (Ω) définie au § 60.

CLASSES (*D*) SÉPARABLES.

51. Un ensemble E est séparable s'il existe un sous-ensemble dénombrable N de E tel que $E < N + N'$. Si l'on prend pour E l'ensemble de la classe, N' appartiendra à E, donc :

Une classe (*D*) *est séparable* s'il existe une suite dénombrable N d'éléments de la classe telle que tout élément appartienne à N ou soit élément-limite d'une suite extraite de N. Ou encore s'il existe une suite dénombrable N d'éléments de la classe telle que la borne inférieure des distances d'un élément de la classe aux différents éléments de N reste nulle.

En la considérant comme une classe (V), on peut choisir les familles de voisinages des différents éléments d'une classe (D) séparable, de sorte que *l'ensemble de tous les voisinages de tous les éléments soit dénombrable.* Il suffit de prendre comme voisinages les sphéroïdes ayant comme centre un des éléments de N et comme rayon l'un des inverses $\frac{1}{n}$ des entiers successifs. On prendra comme voisinages d'un élément a ceux de ces sphéroïdes dont le centre est à une distance de a inférieure au rayon (XXXII, § 16 $^{\text{bis}}$).

On en déduit que des ensembles disjoints dont les intérieurs en sont pas nuls peuvent être dénombrés. Plus généralement :

Soit F une famille d'ensembles G distincts dont chacun possède au moins un élément intérieur ; s'il n'existe aucun élément intérieur à la fois à une infinité non dénombrable d'ensembles G distincts, la famille F est elle-même dénombrable ; la réciproque est d'ailleurs évidente.

Dans une classe (D) séparable : tout ensemble est séparable et condensé.

Par conséquent, on peut répéter ici, en supprimant pour un ensemble la condition d'être séparable ou condensé, tous les théorèmes établis pour les classes (V), (H), (L), (S), (D) où cette condition se trouve imposée :

Un ensemble quelconque E est toujours compris dans la somme d'un de ses sous-ensembles dénombrables N et de son dérivé N' : lorsque E est fermé il est la somme de N et de N'; lorsque E est dense en soi, il appartient à N'; lorsque E est parfait, $E{=}N'$.

L'ensemble des éléments isolés d'un ensemble est dénombrable ; en particulier : tout ensemble isolé est dénombrable.

Tout ensemble a au plus la puissance du continu. Tout ensemble fermé F est la somme d'un ensemble dénombrable N et d'un ensemble parfait P. L'ensemble parfait P peut être caractérisé comme l'ensemble des éléments de condensation de F ou l'ensemble commun à tous les ensembles dérivés de F d'ordres finis ou transfinis. Il y a d'ailleurs un rang α, fini ou transfini à partir duquel tous les dérivés de F sont identiques à P. L'ensemble N peut être représenté comme la somme d'une suite dénombrable d'ensembles dénombrables qui ont une signification simple :

$$N = (F - F') + (F' - F'') + \ldots . = \sum_{n=0}^{n=\alpha} \left[F^{(n)} - F^{(n+1)} \right].$$

$$\text{avec } F^{(\alpha)} = F^{(\alpha+1)} = \ldots\ldots = P.$$

Tout sous-ensemble fermé F d'un ensemble dense en soi G peut être considéré comme le dérivé d'un ensemble dénombrable d'éléments de G.

Tout sous-ensemble fermé F d'un ensemble quelconque E peut être obtenu en supprimant de E les éléments appartenant à un certain ensemble dénombrable de sphéroïdes. Tout ensemble possède la propriété de Lindelöf.

Quel que soit ε il existe une famille dénombrable K_ε de sphéroïdes k_ε de rayon $< \varepsilon$ tel que tout élément soit intérieur à l'un des k_ε.

Etant donnée une famille F de fonctionnelles bornées et également continues en tout élément d'un ensemble quelconque E, sur E, il existe une suite de fonctionnelles extraite de F qui converge sur E vers une fonctionnelle continue en chaque élément de E, sur E. Et la convergence est uniforme sur tout sous-ensemble de E qui est compact et fermé.

PROPRIÉTÉS DE QUELQUES CLASSES IMPORTANTES EN ANALYSE.

Espace euclidien à n dimensions.

52. Les éléments de l'espace euclidien à n dimensions sont définis chacun par l'ensemble de n nombres réels, pris dans un ordre déterminé, nombres qu'on appelle les coordonnées de l'élément ou point considéré. Dans cet espace les éléments d'accumulation sont définis par l'intermédiaire d'une distance, la distance (x, x') de deux points x, x' dont les coordonnées respectives sont $x_1, x_2, \ldots . x_n$ et $x_1', \ldots . x_n'$ étant par définition égale à

$$\sqrt{(x_1 - x_1')^2 + \ldots . + (x_n - x_n')^2}$$

Cet espace constitue une classe (D) séparable, compléte, continue et où deux éléments quelconques d'un sphéroïde peuvent être joints par un arc des Jordan (un segment de droite) situé dans le sphéroïde.

La condition nécessaire et suffisante pour qu'un ensemble soit compact est, dans cet espace, que cet ensemble soit borné, c'est à dire que les coordonnées de tous les points de cet ensemble soient comprises entre deux nombres fixes, ou encore, si l'on préfére, que tous les points de l'ensemble appartiennent à un même sphéroïde. L'espace euclidien est donc évidemment la somme d'une infinité dénombrable d'ensembles compacts.

Une fonctionnelle dont l'argument est un point de cet espace est une fonction de n variables numériques : les coordonnées de cet espace. Une fonctionnelle continue en x^0 sur E est une fonction de n variables qui est continue par rapport à chaque variable $x_1^0 \dots x_n^0$ prise séparément. Mais la réciproque n'est pas toujours vraie.

Classe des fonctions continues.

53. Dans la classe ayant pour éléments les fonctions uniformément continues dans un intervalle déterminé fixe (a, b), appelons suite convergeant vers f une suite de fonctions uniformément continues dans (a, b)

$$f_1(x), \; f_2(x), \dots f_n(x) \dots$$

qui convergent uniformément dans (a, b) vers la fonction $f(x)$ nécessairement uniformément continue dans (a, b). Un élément de la classe est élément d'accumulation d'un ensemble s'il est élément-limite d'une suite convergente d'éléments distincts appartenant à l'ensemble.

On obtient une définition équivalente par l'intermédiaire d'une distance, en appelant distance (f, g) de deux éléments f, g, le maximum de $| f(x) - g(x) |$ dans (a, b). On pourrait évidemment remplacer cette définition par une autre n'altérant ni la convergence des suites ni leur limite, mais il est évident que cette définition offre un caractére de simplicité qui en impose le choix.

La classe considérée est une classe séparable compléte, continue (XVIII, § 56) et deux quelconques de ses éléments appartenant à un même sphéroïde peuvent être joints par un arc de Jordan appartenant à ce même sphéroïde, (XXXII, § 32).

La condition nécessaire et suffisante pour qu'un ensemble de fonctions uniformément continues dans (a, b) forme dans cette classe un ensemble compact est que les fonctions de cet ensemble soient bornées et également continues sur (a, b), (XVIII, § 57). On en conclut

(XIX, § 28))que cette classe ne peut être considérée comme la somme d'un ensemble dénombrable d'ensembles compacts.

Le fait que cette classe est séparable peut s'exprimer comme je l'ai montré dans ma Thèse (XVIII, § 56) sous une forme analytique assez saisissante.

On peut former une fois pour toutes une série de fonctions continues

$$v_1(x) + v_2(x) + \ldots. + v_p(x) + \ldots.$$

telle que toute fonction continue f puisse être representée comme somme de cette série quand on y procède préalablement à un groupement convenable de ses termes :

$$f(x) = \left[v_1(x) + \ldots. + v_{p_1-1}(x)\right] + \left[v_{p_1}(x) + \ldots. + v_{p_2-1}(x)\right] + \left[v_{p_2}(x) + \ldots.\right] + \ldots.$$

et ceci avec convergence uniforme sur tout intervalle fini où f est uniformément continue. On peut même supposer que les $v_n(x)$ sont des fonctions particulièrement simples, par exemple que ce sont des polynomes à coefficients rationnels ou bien que leurs représentations graphiques sont des lignes polygonales dont les sommets ont des coordonnées rationnelles.

Six ans après, Sierpinski démontrait dans le Bulletin de l'Académie des Sciences de Cracovie (1912, page 86) une proposition analogue où le groupement des termes est remplacé par un changement de l'ordre des termes.

54. On peut aussi noter que les propriétés des classes (D) énoncées au § 47 en ce qui concerne la puissance des familles K_ϵ répondent entièrement dans le cas plus général des classes (D) à une question posée en 1897 par M. Hadamard dans le cas spécial de la classe des fonctions continues (XXIX).

55. Si l'on voulait considérer le cas des fonctions continues pour toutes valeurs de x, on serait amené à considérer comme suite convergente une suite de fonctions continues qui converge uniformément *dans tout intervalle fini*. Alors on aurait encore une classe (D) séparable, parfaite, complète. On pourrait adopter comme définition de la distance (f, g) la définition proposée par Gateaux, savoir la borne inférieure quand le nombre positif α varie, de $\frac{1}{\alpha} + (f, g)_\alpha$, la quantité $(f, g)_\alpha$ désignant le maximum de $/ f(x) - gx) /$ dans l'intervalle $(-\alpha, +\alpha)$.

On pourrait aussi en s'inspirant de la définition que j'avais proposée dans ma Thèse pour le cas des fonctions holomorphes, adopter comme valeur de la distance (f, g), l'expression

$$\frac{(f, g)_1}{1+(f, g)_1}+\ldots\ldots+\frac{1}{n!}\,\frac{(f, g)_n}{1+(f, g)_n}+\ldots\ldots$$

Classe des arcs de courbes continues.

56. Appelons arc de courbe continue AB une suite ordonnée continue de points de l'espace à trois dimensions ayant comme origine A et extrémité B. Tout point de la courbe est déterminé par : sa position dans l'espace et son rang. De sorte qu'un même point de l'espace peut être le siége de plusieurs points de la courbe de rangs distincts : ce sera un point multiple. On exclut bien entendu le cas où tous les points de la courbe dont les rangs sont situés entre deux rangs distincts coïncideraient dans l'espace. Mais nous supposerons en outre que l'ordre de multiplicité des points multiples est toujours dénombrable.

En disant que l'arc est continu, on entend que pour tout point P de la courbe de rang α, et tout nombre $\epsilon>0$, il existe deux rangs comprenant α : $\beta<\alpha<\gamma$ tels que tout point de la courbe de rang compris entre β et γ soit à une distance de P inférieure à α. (Dans le cas où P coïnciderait avec A ou B, on supposerait $\beta=\alpha$, ou $\gamma=\alpha$, respectivement).

Nous avons ainsi donné de l'arc de courbe continue une définition purement géométrique où n'intervient aucune représentation analytique. On démontre d'ailleurs (XIX, § 35) que cette définition est équivalente à la suivante :

Un arc de courbe continue AB est l'image d'un segment de droite dont les extrémités correspondent à AB. Ou encore, ce qui revient au même.

Un arc de courbe est une suite de points dont les coordonnées peuvent être représentées sous la forme

$$x=f(t),\ y=g\ (t),\ z=h(t)$$

où f, g, h sont trois fonctions de t uniformément continues dans l'intervalle (0, 1) (par exemple) et non constantes à la fois dans un même intervalle, l'ordre des points de l'arc étant celui qu'on obtient en faisant croîte t de 0 à 1.

Etant donnée une représentation analytique de l'arc AB toutes les représentations analytiques du même arc s'obtiennent par la substitution

$$t = \theta(t')$$

où $\theta(t')$ est une fonction continue qui croit constamment de 0 à 1 quand t' croit de 0 à 1 (XVIII, § 77).

57. Ceci étant, la classe considérée a pour éléments les arcs de courbe continue et les éléments d'accumulation y sont définis par l'intermédiaire de la définition de l'élément-limite d'une suite convergente d'arcs continus

On dira qu'une suite d'arcs A_1B_1, $A_2B_2, \ldots A_nB_n$ ..est une suite convergente qui a l'arc AB pour élément-limite si l'on peut établir une correspondance biunivoque et bicontinue (conservant l'ordre des points) telle que si M_n est le point de $A_n B_n$ correspondant au point M de AB, la distance MM_n converge *uniformément* vers zéro.

Il revient au même de dire qu'il existe une représentation analytique de l'arc AB

$$x = f(t),\ y = g(t),\ z = h(t)$$

et une représentation analytique de l'arc A_nB_n

$$x = f_n(t),\ y = g_n(t),\ z = h_n(t)$$

telles que $/f_n - f/$, $/g_n - g/$, $/h_n - h/$ convergent uniformément vers zéro

Il est maintenant important de remarquer que dans la classe que nous venons de définir, on peut définir les éléments d'accumulation par l'intermédiaire d'une distance.

J'ai proposé dans ma Thèse la définition suivante (XVIII, § 78) Etant donnés deux arcs AB, A_1B_1, on établit entre eux une correspondance S biunivoque, bicontinue et conservant l'ordre des points ; soient M, M_1 les points correspondants. Appelons δ_s le maximum de la longueur MM_1 lorsque M parcourt AB. On appellera distance des deux arcs donnés la borne inférieure de δ_s quand la correspondance S varie de façon quelconque.

On voit alors que la classe des arcs continus est une classe (D) séparable, complète et continue (XVIII, § 79 à 84) (D'après la section précédente, la classe est évidemment l'ensemble dérivé de l'ensemble dénombrable des lignes polygonales dont les sommets ont des coordonnées rationnelles). En outre, dans cette classe, on peut joindre deux éléments appartenant à un même sphéroïde par un arc de Jordan appartenant à ce sphéroïde (XXXII, § 32).

Pour qu'un ensemble d'arcs continus forme un ensemble compact, il faut et il suffit que les arcs de cet ensemble soient tous situés dans un même domaine fini (par exemple une sphère fixe) et soient uniformément divisibles (XVIII, § 91).

Ces deux conditions sont indépendantes de la représentation analytique ; on dit en effet que des courbes sont uniformément divisibles si à tout nombre $\epsilon > 0$ on peut faire correspondre un entier n, tel que chacune des courbes considérées puisse être divisée en n arcs dont l'oscillation (la plus grande corde) soit inférieure à ϵ.

Traduit en langage analytique, ceci veut dire qu'il existe *un* système de représentations analytiques simultanées des différentes courbes de la famille où les fonctions qui représentent les coordonnées sont bornées dans leur ensemble et également continues (XVIII, § 38).

Comme application : un ensemble de courbes rectifiables situées dans un domaine fini est compact si leurs longueurs sont bornées dans leur ensemble. Mais cette condition suffisante n'est pas nécessaire (XVIII § 93). Il n'est même pas nécessaire pour qu'un ensemble d'arcs soit compact, que ces arcs soient rectifiables.

Pour démontrer les diverses propositions précédentes, il est utile d'employer le théorème suivant démontré dans la Note I de ma Thèse (XVIII, page 67).

Etant donné sur une droite ox, un ensemble G d'intervalles I sans points communs dans le segment fondamental $(0 \leq x \leq 1)$, il existe au moins une fonction continue qui va sans jamais décroître de 0 à 1 quand x croit de 0 à 1 et qui n'est constante que dans les intervalles I.

J'avais cité comme cas particulier, le cas où l'ensemble complémentaire de l'ensemble des points des intervalles I est un certain ensemble de mesure nulle considéré par *G.* Cantor (XVIII, § 99), Cinq ans après, W. Sierpinski a établi à son tour le même théorème d'existence dans ce cas particulier, en montrant en outre que sous certaines conditions fonctionnelles la solution est unique (Bulletin de l'Académie des Sciences de Cracovie, 1911, page 577).

Classes des fonctions holomorphes.

58. Cette classe est formée des fonctions holomorphes *à l'intérieur* d'une aire fixe A. On y considère une suite d'éléments $f_1(z)$, $f_2(z)$,... $f_n(z)$,....comme convergaent vers un élément $f(z)$ si $f_n(z)$ converge vers $f(z)$ uniformément dans toute aire complétement intérieure à A. Un

élément est alors élément d'accumulation d'un ensemble s'il est élément-limite d'une suite d'éléments distincts appartenant à l'ensemble.

Il faut remarquer qu'il aurait pu paraître plus naturel de supposer que chaque élément est une fonction non seulement holomorphe à l'intérieur de A, mais aussi holomorphe ou tout au moins définie et continue sur A et de supposer que la convergence uniforme considérée aie lieu uniformément dans tout A contour compris. Mais alors on aurait été amené à exclure pour A le cercle de convergence d'une fonction holomorphe, par exemple ou à ne pas considérer celle ci dans son cercle de convergence comme l'élément limite des sommes des termes de son développement de Taylor.

Non seulement les définitions proposées préviennent cette exclusion, mais elles permettent de définir aussi les éléments limites par l'intermédiaire d'une "distance." J'ai proposé dans ma Thèse (XVIII, § 70) comme définition de la distance de deux fonctions $f(z)$, $g(z)$ holomorphes à l'intérieur d'une aire A la quantité

$$(f, g) = \frac{(f, g)_{A_1}}{1 + (f, g)_{A_1}} + \ldots\ldots + \frac{1}{n!} \cdot \frac{(f, g)_{A_n}}{1 + (f, g)_{A_n}} : \ldots\ldots$$

où $(f, g)_{A_n}$ désigne le maximum de $/f(z) - g(z)/$ dans une aire A_n entièrement intérieure à A, $(A_1, A_2, \ldots . A_n$ ..étant une suite d'aires chacune comprise dans la suivante et dans A et dont la somme embrasse A).

Bien entendu, d'autres définitions de la distance conduiraient aux mêmes éléments d'accumulation et cette définition n'offre pas le même caractère de simplicité que celle adoptée dans la classe des fonctions continues. Il serait intéressant d'en trouver une équivalente et plus simple, telle par exemple que l'on ait l'égalité

$$(\lambda\ (f-g).\ \lambda'\ (f-g)\ ) = \lambda - \lambda' \quad (f, g)$$

quelles que soient les constantes λ, λ' et les fonctions $f(z)$, $g(z)$ holomorphes dans A.

Mais l'essentiel est le fait que la classe est une classe (D) qui est d'ailleurs en outre séparable, complète et continue (XVIII, § 7). Un des ensembles dénombrables d'éléments dont la classe est le dérivé est l'ensemble simple constitué par les polynomes à coefficients rationnels (la partie réelle et la partie imaginaire séparément rationnelles).

En outre si l'on adopte la définition de la distance mentionnée plus haut, on peut joindre deux éléments quelconques de cette classe appartenant à un même sphéroïde par un arc de Jordan appartenant à ce sphéroïde (XXXII, § 32).

Dans la classe considérée, la condition nécessaire et suffisante pour qu'un ensemble soit compact est que les fonctions de cet ensemble restent en module inférieures à un nombre fixe dans tout aire intérieure à *A* (XVIII, § 73). On en conclut que cette classe ne peut être décomposée en une infinité dénombrable d'ensembles compacts.

Remarque I. On peut ce même étudier les ensembles compacts de fonctions meromorphes, etc......

II. Plusieurs années aprés ma Thése, s'est repondu l'usage d' employer l'expression de famille normale, pour ensemble compact, dans le ces où les élements sont des fonctions analytiques.

Espace E_ω.

59. Les éléments de cet espace sont chacun définis par une suite infinie de nombres réels appelés coordonnées de l'élément ou point. Un point x de l'espace E_ω est dit limite d'une suite convergente de points $x^{(1)}, x^{(2)}, \ldots x^{(p)}, \ldots$ si les coordonnées des points de cette suite tendent respectivement vers les coordonnées de même rang du point x. Un point x est point d'accumulation d'un ensemble E s'il est limite d'une suite convergente de points distincts appartenant à E.

Ainsi dans cet espace, c'est la notion d'élément limite d'une suite convergente qui est fondamentale; elle est d'ailleurs toute naturelle et s'impose pour l'étude d'un certain nombre de questions.

Il est cependant remarquable qu'on peut obtenir une définition équivalente par l'intermédiaire d'une distance. J'ai proposé (XVIII, § 62) l'expression

$$(x, x') = \frac{|x_1 - x_1'|}{1 + |x_1 - x_1'|} + \ldots + \frac{1}{n!}\,\frac{|x_n - x_n'|}{1 + |x_n - x_n'|} + \ldots$$

pour la distance de deux points x, x' dont les coordonnées respectives sont $x_1, x_2, \ldots x_n \ldots$; $x_1', x_2', \ldots x_n' \ldots$. Mais bien entendu d'autres expressions conviendraient tout aussi bien. L'essentiel, c'est le fait que l'espace E_ω est une classe (D), séparable, compléte, continue. „Le fait qu'on peut y joindre deux points appartenant à un même sphéroïde par un arc de Jordan appartenant à ce même sphéroïde fait intervenir essentiellement la définition particulière de la distance choisie"; mais nous avons vu qu'il a des consequences indépendantes du choix particulier de cette distance.

Dans cet espace la condition nécessaire et suffisante pour qu'un ensemble soit compact est que les coordonnées des points de l'ensemble soient bornées pour chaque rang (XVIII, § 66). Cet espace n'est pas la somme d'une infinité dénombrable d'ensembles compacts.

Espace Ω.

60. Les éléments de l'espace Ω sont chacun défini par une suite infinie de nombres réels (appelés coordonnées de l'élément) et dont la somme des carrés converge. Dans cet espace les éléments d'accumulation sont définis par l'intermédiaire d'une distance, la distance (x, x') de deux points dont les coordonnées respectives sont $x_1, x_2, \dots x_n, \dots$ et $x_1', x_2', \dots x_n' \dots$ étant par définition égale à

$$\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + \dots + (x_n - x_n')^2 + \dots}$$

La série sous le radical est certainement convergente puisque
$(x_1 - x_1')^2 + \dots + (x_n - x_n')^2 \leqq 2(x_1^2 + \dots + x_n^2) + 2(x_1'^2 + \dots + x_n'^2)$

Pour qu'une suite de points de l'espace $\Omega \cdot x^{(1)}, x^{(2)}, \dots x^{(p)} \dots$ converge vers un point x, il faut que les coordonnées tendent respectivement vers les coordonnées de même rang de x. Mais cette condition, n'est pas suffisante.

Cet espace constitue une classe (D) séparable, compléte, continue, (XXXVI) et où deux éléments quelconques d'un sphèroïde peuvent être joints par un arc de Jordan (un segment de droite) situé dans le sphéroïde (XXXIII, § 32). J'ai d'ailleurs montré (XXXVI) qu'on peut y développer une géométrie projective et métrique entiérement semblable à celle de l'espace euclidien à un nombre fini de dimensions.

Dans cet espace, la condition nécessaire et suffisante pour qu'un ensemble soit compact peut s'exprimer sous diverses formes (XIX, page 18, § 29). O peut lui donner la suivante :

la somme des carrès des coordonnées des points de l'ensemble doit être bornée sur l'ensemble et y converger uniformément vers sa limite.

O en conclut facilement (XIX, § 31) qu'un tel espace ne peut être décomposé en une infinité dénombrable d'ensembles compacts.

Classe des fonctions mesurables.

61. On a intérét parfois en Analyse à considérer, au lieu de la classe des fonctions continues, la classe (M) plus étendue des fonctions mesurables. Appelons classe (M) la classe dont les éléments sont des

fonctions mesurables au sens de M. Lebesgue et où une suite d'éléments est dite convergente quand elle converge "en mesure" au sens de F. Riesz. On dit que $f_n(x)$ converge en mesure vers $f(x)$ dans l'intervalle fixe (a, b) si quel que soit $\eta > 0$, il existe un nombre $\epsilon > 0$, et un entier p tels que pour $n > p$

$$|f_n(x) - f(x)| < \eta$$

sauf peut être dans un ensemble de points de l'intervalle (a, b) pouvant être enfermé dans un ensemble d'intervalles de longueur totale $< \epsilon$. On remarquera que cet ensemble pouvant varier quand n varie, il n'en résulte pas que f_n converge presque partout vers f; il peut même arriver que $f_n(x)$ ne converge nulle part vers f. Mais pour chaque valeur de n, $f_n(x)$ ne diffère sensiblement de $f(x)$ que sur un ensemble dont la mesure tend vers zéro avec $\frac{1}{n}$.

62. Ceci étant, on peut démontrer que la classe (M) des fonctions mesurables est une classe (D). Autrement dit la convergence en mesure de F. Riesz peut être définie par l'intermédiaire d'une définition convenable de la distance de deux fonctions mesurables. J'ai proposé (XXXVII) comme expression de la distance de deux fonctions $f(x)$, $\phi(x)$ mesurables au sens de M. Lebesgue sur l'intervalle (a, b), la borne inférieure de la somme

$$\omega + m_{|f-\phi|>\omega}$$

lorsque ω prend toutes les valeurs positives ou nulles, en désignant par $m_{|f-\phi|>\omega}$ la mesure de l'ensemble mesurable des points x où

$$|f(x) - \phi(x)| > \omega$$

Il est probable qu'on pourrait concevoir une définition de la distance qui fournisse encore la convergence en mesure et qui serait pourtant moins artificielle. Mais l'essentiel est que cette définition existe et que par conséquent on puisse étendre immédiatement aux fonctions mesurables toutes les propriétés des classes (D) quand la convergence n'est imposeé "qu'en mesure." Il y a lieu d'observer qu'il n'en serait plus de même si on remplaçait celle-ci par la convergence "presque partout" de M. Lebesgue. Celle-ci, à peine plus restrictive, puisqu'elle n'exige la convergence ordinaire qu'à l'exception eventuelle d'un ensemble fixe de mesure nulle n'est pourtant pas compatible avec une definition de la distance (XXIX).

D'ailleurs une telle classe (D) est séparable; c'est même l'ensemble dérivé de l'ensemble dénombrable des fonctions qui sont constantes et de valeurs rationnelles dans chacune des subdivisions de l'intervalle (a, b) limitées par un nombre fini variable de points d'abscisses rationnelles.

Cette classe est aussi complète et continue ; et on peut joindre deux de ses éléments appartenant à un même sphéroïde par un arc de Jordan appartenant à ce sphéroïde (XXXII, § 32, VII). Il serait intéressant d'établir à quelle condition un ensemble d'éléments de cette classe est compact.

Classe D_ω.

63. Appelons classe D_ω la classe dont chaque élément x est déterminé par une suite infinie de nombres réels

$$x_1, x, \ldots . x_n \ldots .$$

qu'on peut appeler les coordonnées de rangs 1, 2,....n, ...du poin- x, et où une suite d'éléments ou points $x^{(1)}$, $x^{(2)}$,....$x^{(\nu)}$,....est ditet convergente vers le point x si les coordonnées de $x^{(\nu)}$ tendent *uniformément* vers les coordonnées de même rang de x.*

On voit que les trois classes E_ω, Ω, D_ω sont trois espaces à une infinité dénombrable de coordonnées. Dans les trois espaces pour qu'une suite de points $x^{(1)}$, $x^{(2)}$....$x^{(p)}$....converge vers un point x, il faut que les coordonnées de $x^{(\nu)}$ convergent vers les coordonnées de même rang de x. Mais, alors que cette condition est suffisante pour E_ω, il faut qu'elle aie lieu uniformément pour D_ω ; cette condition supplémentaire suffisante pour D_ω, n'est que nécessaire pour Ω.

Ces différences sont essentielles comme le montre la remarque suivante : si les classes E_ω et Ω sont, comme nous l'avons vu, séparables, il n'en est pas de même de D_ω, (VI, page 163). Et la remarque suivante accentue le caractère de plus grande généralité de D_ω : toute classe (D) séparable, complète et parfaite est homéomorphe d'une partie de D_ω ; plus encore on peut établir entre cette classe et une partie de D_ω une correspondance qui conserve les distances (XIX, page 12, § 23). Dans ce but, on appelera a_0, a_1, a_2, ...a_n....un ensemble dénombrable d'éléments de la classe (D) considérée, tel que tout élément de la classe appartienne à cette suite ou à son ensemble dérivé. Il suffira, alors de faire correspondre à tout élément a de la classe (D), le point x de coordonnées

$$x_1 = (a_1, a) - (a_1, a_0) \,;\; x_2 = (a_2, a) - (a_2, a_0), \ldots . , x_n = (a_n, a) - (a_n, a_0), \ldots .$$

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* J'appelai précédemment (VI, XIX) classe D la classe actuelle, ce qui prètait ω confusion avec les classes (D) du § 46.

Malgré tout, la classe D_ω est relativement simple, puisque c'est une classe (D) compléte et continue.

La définition de la distance qui fournit une définition des suites convergentes équivalente à la définition indiquées plus haut, s'impose ici ; la distance de deux points x, x' sera la borne supérieure des valeurs absolues des différences $/x_n - x_n'/$ des coordonnées de même rang de x et de x'. Toutefois, si l'on veut éviter des distances infinies, il sera préférable de n'admettre comme point de D_ω qu'un point dont les coordonnées sont bornées dans leur ensemble.

Ceci admis, deux points appartenant au même sphéroïde peuvent encore être joints par un arc de Jordan appartenant à ce sphéroïde (XXXII, § 32).

Remarque finale.

64. En terminant ce sommaire des principes de l'Analyse fonctionnelle, il convient d'observer qu' à résumer (sans rappeler les démonstrations) un grand nombre de travaux, on encourt facilement le risque de commettre des erreurs, par l'oubli d'une condition essentielle dans un énoncé. Je serais donc reconnaissant aux mathématiciens qui voudront bien me signaler les énoncés inexacts que j'aurais insérés par mégarde et aussi les résultats nouveaux rentrant dans le cadre de ce mémoire et dont je n'aurais pas eu connaissance. L'Analyse fonctionnelle fait constamment de nouveaux progrès et je me propose de développer à nouveau cette premierè ébauche en profitant des observations qui m'auront été faites.

Bien que de nombreuses propriétés démontrées dans ma Thèse pour certaines classes aient pu être étendues depuis lors à des classes plus générales, et mentionnées ici sous cette nouvelle forme, il n'est pas douteux qu'un certain nombre de résultats mentionnés dans le présent travail puissent être aussi étendus à leur tour. J'en ai signalé quelques uns pour lesquels cette extension parait certaine. Mais, en classant par, ordre de généralité les résultats acquis, le présent travail permetra sans doute d'apercevoir la possibilité de nouvelles extensions et aidera peut être à les réaliser.

D'autre part, j'ai été amené à modifier quelque peu dans la suite de mes travaux les notations et la terminologie proposées dans ma Thèse. Celles-ci ayant été utilisées par différents auteurs, je serais reconnaissant à ceux d'entre eux qui voudront encore me faire l'honneur de s'en servir, d'employer de préférence cette terminologie et ces notations dans le sens proposé dans le présent mémoire.

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EQUILIBRIUM IN THE FRACTIONAL PRECIPITATION OF SILVER CHLORIDE AND SILVER BROMIDE.

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From a mixture of the solutions of two binary salts, having a common ion, if the non-common ions are fractionally precipitated by addition of an insufficient quantity of a third salt, then, the more sparingly soluble component of the non-common ions is found to preponderate in the mixed precipitate. The relation between the composition of such mixed precipitate and that of the residual solution standing in equilibrium with them, and also, the influence of the degree of insolubility of the component in the precipitate upon the state of equilibrium are the subjects of the present investigation.

From a mixture of KCl and KBr solution, AgCl and AgBr were fractionally precipitated by addition of insufficient quantity of $AgNO_3$. The composition of the halogens in the precipitate and those remaining in the solutions were first studied with a view to find out a definite relation between them.

A series of preliminary experiments were performed to find out whether freshly precipitated AgCl could be completely converted into AgBr by means of KBr solution, just sufficient for the transformation.

To about 0·7 gm. of freshly precipitated AgCl, in an amber coloured bottle 50 c.c. of decinormal solution of KBr was added and the mixture was vigorously shaken in a shaking machine for definite periods of time which were noted. The mixed precipitate was transferred to a Gooch crucible, dried and weighed. In the Table I the figures of the column V give the ratio of AgCl to the mixed halogen precipitate by weight, the ratio of complete transformation being 1·3101.

Table I.

No.	Time of shaking.	Wt. of AgCl ppt. in A.	Vol. of halogen sol.	Wt. of mixed ppt. in B.	The Ratio B : A.
I.	2 min.	0·6873 gm.	50 c.c. KBr 0·101 N	0·8830 gm.	1·2847
II.	5 ,,	0·6889 ,,	,,	0·8906 ,,	1·2927
III.	15 ,,	0·6883 ,,	,,	0·8940 ,,	1·2987
IV.	30 ,,	0·6889 ,,	,,	0·8963 ,,	1·3010
V.	75 ,,	0·6869 ,,	,,	0·8959 ,,	1·3031
VI.	24 hours	0·6560 ,,	,,	0·8565 ,,	1·3056
VII.	,,	0·9390 gm. Ag Br ppt.	50 c.c. NaCl 0·1 N	Wt. of mixed ppts. 0·9350	The ratio B : 1·3059
VIII.	,,	0·3613 gm. AgCl ppt.	25 c.c. KBr 0·2 N	0·4727	1·3089

Thus, it is evident from the figures of the table that complete transformation of AgCl to AgBr does not take place, even in presence of sufficient KBr. But it appears that the system attains a state of equilibrium which is identical when AgBr is similarly treated with NaCl solution (cf. exp. VII, Table I).

The work done by Küster* towards the elucidation of such equilibrium in Silver Halogen precipitate is important. He conducted a series of experiments, which are recorded in the Table II, by fractionally precipitating a mixture of KCl and KBr solution with an insufficient amount of $AgNO_3$ solution. The total concentration of Halogens in the solution was always the same in different experiments and after treatment with $AgNO_3$ the residual solution was nearly Normal with regard to the former. The volume of the solution was always 1000 c.c. The temperature of the experiment was 19° ± 1°C., the time of shaking the mixture was 2 to 12 hours.

Some of his experiments were repeated with a slight modification. Owing to slight solubility of AgCl in strong KCl solution, the amount of silver precipitated was always less than the amount added to the solution; to avoid this discrepancy, in the repetition of Küster's experiments, the solution previous to the addition of $AgNO_3$ was carefully treated with dilute $AgNO_3$ solution till a slight opalescence persisted. These "treated" solutions were found to precipitate $AgNO_3$ almost completely. The experiments were conducted at 25° ± 1°C and the time of shaking the mixture was ever 24 hours. The concentration

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* *Zeit. anorg. Chem*, *23*, (1898).

Table II.

No.	KBr sol.	KCl sol. 2N.	Ag NO$_3$ sol. 0·1 N	Water (+10 c.c. HNO$_3$)	KBr in mili-mol.	KCl in mili-mol.	Halogens in precipitate in mili-mol.		Solution.		Ratio.		Ration.	
							AgBr	AgCl	KBr in mili. mol.	KCl in mili-mol.	$\frac{AgBr}{AgCl}=K_1$	$\frac{KBr}{KCl}=K$ × 10^{-3}	$\frac{K_1}{K_2}=\pi$.	AgBr. / KBr.
1	10 c.c.	505 c.c.	100 c.c.	385 c.c.	0·1	1010	0·064	9·84	0·036	1000	0·0065	·036	184	1·81
2	20 ,,	,,	,,	375 ,,	0·2	,,	0·131	9·77	0·069	,,	0·0134	·069	194	1·9
3	30 ,,	,,	,,	365 ,,	0·3	,,	0·207	9·70	0·093	,,	0·0213	0·093	228	2·2
4	50 ,, $\frac{N}{100}$	,,	,,	345 ,,	0·5	,,	0·349	9·55	0·151	,,	0·0365	0·153	241	2·32
5	70 ,,	,,	,,	325 ,,	0·7	,,	0·503	9·40	0·197	,,	0·0535	0·197	271	2·55
6	100 ,,	,,	,,	295 ,,	1·0	,,	0·726	8·18	0·274	1001	0·0791	0·274	289	2·65
7	20 c.c.	,,	,,	375 ,,	2	,,	1·44	8·47	0·56	1002	0·17	0·563	301	2·55
8	30 ,, $\frac{N}{10}$	,,	,,	365 ,,	3	,,	2·14	7·77	0·86	,,	0·275	0·861	320	2 48
9	50 ,,	500 c.c.	,,	350 ,,	5	1000	3 49	6·41	1·51	994	0·545	1·52	603	2·31
10	70 ,,	,,	,,	330 ,,	7	,,	4·74	5·17	2·26	995	0·917	2·28	403	2·1
11	100 ,,	,,	,,	300 ,,	10	,,	6·29	3·61	3·71	996	1·74	3·72	469	1·7
12	20 c.c.	495 c.c.	,,	385 ,,	20	990	8·6	1·31	11·4	989	6·58	11·5	570	....
13	30 ,,	490 ,,	,,	380 ,,	30	980	9·17	0·736	20·83	979	12·45	21·3	585	....
14	50 ,, $\frac{N}{1}$	480 ,,	,,	370 ,,	50	960	9·52	0·382	40·48	960	24·9	42·2	591	....
15	100 ,,	455 ,,	,,	345 ,,	100	910	9·75	0·152	90·25	910	63·3	99·2	638	....
16	200 ,,	405 ,,	,,	295 ,,	200	810	9·83	0·072	190·2	810	136·6	234·4	581	....
17	400 ,,	305 ,,	,,	195 ,,	400	610	9 87	0·032	390·1	610	308·5	639·5	482	....

of different salt solutions was slightly different from those of Küster, hence instead of volume their actual concentrations in terms of milimols. are given in the following table. The volume of the solution was reduced from 1000 c.c to 250 c.c. In other respects the experiments were identical with those of Küster.

The results of the experiments are recorded in the following table, in which, R_1, R_2, R'_2 indicate respectively the ratios AgBr : AgCl, KBr : NaCl and Br′ ion : Cl′ ion; and, π, π', π_k are respectively the ratios $R_1 : R_2$, $R_1 : R'_2$, and $K_1 : K_2$ (of Küsters experiment, cf. Table II).

Table III.

Temperature 25° ± 2° C.

Concentration are expressed in mili-mols.

Total volume of the mixture=250 c.c.

No.	KBr	NaCl	$AgNO_3$	$R=\frac{AgBr}{AgCl}$	$R_2 \times 10^{-3} = \frac{KBr}{NaCl}$	$R_2' \times 10^{-3} = \frac{Br'}{Cl'}$	$\frac{R_1}{R_2} = \pi$	$\frac{R_1}{R'_2} = \pi'$	Küster's expt. π_k	Absorbed Ag in mili-mols.
I.	·10	246	2·52	·031	0·1	0·146	310	212	..	·034
II.	·125	246	2·49	·0318	0·16	0·212	218	140	228	·034
III.	·251	246	2·52	·0778	0·31	0·45	255	173	296	·033
IV.	1·258	246	,,	0·473	1·8	2·66	264	178	369	·016
V.	1·762	245	,,	0·805	2·57	3·8	312	212	414	·014
VI.	2·517	,,	,,	1·502	4·2	6·0	376	250	475	·015
VII	7·55	241	,,	9·91	21·8	30·5	455	325	600	·065
VIII.	12·58	236	,,	21·5	43·0	55·2	501	390	610	·007
IX.	25·17	224	,,	55·0	100·7	124·0	545	443	663	·0085
X.	50·35	199	,,	104·0	240·	274·0	433	380	607	·012

The results of the present experiments are different from those of first author as will be evident from a glance at the numerical values of π and π. This difference is appreciably large in experiments V to X (Table III) and is probably due to insufficient time for equilibrium and also to the fact that the quantity of AgCl dissolved in the chloride solution decreases with the rise of Br′ ion concentration in the solution, as shown in the last column of the Table III. In Küster's experiments, however, the total quantity of silver in the precipitate seems to take a mean value, viz. 9·904 mili-mol. which does not alter regularly with Br′ ion concentration, as expected from the above experimental results and the following consideration—

The change in the solubility of AgCl in NaCl solution which maintains a nearly uniform concentration in different experiments is mainly due to the change of Br′ ion concentration (cf. Table VA). If the′ solubility be due to the formation of a complex salt of the type

$$Na_x \, AgCl_{x+1}$$

(x being an unknown integral number) which stands in equilibrium with Ag′ ion and Cl′ ion, according to the following equation :—

$$\frac{Ag' \times (Cl')^x}{AgCl_{x+1}} = \text{Constant}.$$

Then evidently the concentration of the complex salt will be proportional to the product of $Ag' \times (Cl')^x$. The concentration of Cl′ is almost constant in our experiments, while the concentration of Ag′ ion falls with the increase of Br ion concentration in the solution. So the product of $Ag' \times (Cl')^x$ will fall and along with it the solubility of AgCl in the solution. In an analogous way Bodlander and Fittig (*Zeit. Phys Chems. 39*, 605) determined the solubility-product of AgBr in aqueous solution by observing the solubility of AgBr in ammonium hydroxide.

It may be observed in Table II that though K_1 and K_2 vary within wide limits in different experiments, their ratio, viz π is almost constant.

Küster assumes that in the case of AgCl and AgBr when freshly precipitated " each mixture works as a single phase towards the liquid phase staying with them in equilibrium i.e. forms an *isomorphous mixture.*" He however mentions a different conclusion arrived at by Spectator, namely, that an Isomorphous mixture of AgCl and AgBr is either not formed at all or is formed only to a small extent.

Theil (*Zeit. anorg, Chem.*, *24* [1900]) determined the concentration of Ag′ ion in the solution of Küster's experiments, by the measurements, E.M F. of concentration cells. He concludes, " generally, the solubility-product of each salt is constant, so long as the active mass is constant and pure, but with the formation of isomorphous mixture, its active mass alters, consequently the solubility and the solubility-product, from what it is in pure condition. "

Thus supporting Küster's conclusion of formation of isomorphous mixture he further points out that " AgCl and AgBr are completely miscible with each other and the concentration of Ag' ion in the solution will depend upon the composition of the mixed precipitate."

The last column of the Table II shows that the ratio of AgBr to AgCl is fairly constant from experiments III to IX to which Küster ascribes the following reason:

"The concentration ratio of Cl' and Br' ions is always that of ionised AgCl and AgBr, thus:

$$\frac{\text{Undissociated AgBr}}{\text{Undissociated AgCl}} = \frac{\text{Total KBr}}{\text{Total KCl}}\,\cdot$$

or,

$$\frac{\text{Undissociated AgBr}}{\text{Total KBr}} = \frac{\text{Undissociated (AgCl and AgBr)}}{\text{Total KBr and KCl}}\,\cdot$$

The right hand side of the equation is constant when the precipitate contains almost wholly AgCl, hence the left hand side also. That is, the partition ratio of AgBr between precipitate and solution is nearly constant (cf. exp. III to IX, Table II).

As a matter of fact, the AgBr is assumed to be completely dissociated to Ag' and Br' ions (its solubility being extremely small). Hence the concentration of "undissociated" AgBr is quite an indefinite quantity and inadequately chosen to explain the phenomenon. Again, from this explanation it is natural to expect an identical relation existing in the case of AgCl in experiments containing mixed precipitate poor in AgCl. From the results of the experiments this is however not possible to show.

On the other hand, if such complete miscibility be possible in the mixed precipitate the abnormal values of π in experiments I to VIII (Table II) stand unexplained.

Hence the existence of isomorphous mixture of AgCl and AgBr is quite doubtful, as pointed out by Speckator.

Let us suppose that in Küster's experiments, AgCl is first precipitated, and in the solution the product:

$$\mathrm{Ag}' \times \mathrm{Cl}' = \text{Constant.}$$

which is equal to solubility-product (*SP*) of AgCl. In the presence of Br' ion) however, the concentration of Ag' ion is too high since the *SP* of AgBr is about 1/300 of the *SP* of AgCl. The concentrations of Ag', Cl' and Br' ions therefore readjust themselves in a *definite* way, since the same ultimate condition is possible if AgBr is first precipitated (Cf. Expt. VI and VII, Table I).

Thus it will be interesting to study the equilibrium of fractional precipitation, from the stand point of variation of Ag·ion with the change of concentration of Br′ and Cl′ and also the change of the ratio AgCl to AgBr in the precipitate.

The concentration of Ag′ ion in different experiments (after HNO_3 was carefully neutralised with Na_2CO_3) were determined from E.M F. of the respective concentration cells.

In the following table the concentrations are given in gram-mols. per litre and the symbol *P* represents the product of concentration of the three ions viz. Ag′ × Cl′ × Br′.

Table IVa.

Concentrations are given in gram-mols. per litre.

Exp. No.	$KBr \times 10^{-3}$	Cl′ ion	Br′ ion $\times 10^{-3}$	Ag· ion $\times 10^{-10}$	$P \times 10^{-13}$	$\frac{Br'}{Cl'} = R'_2 \times 10^{-3}$
III.	1·004	0·675	0·3	3·35	0·68	0·45
IV.	5·032	0·673	1·78	3·09	3·7	2·66
V.	7 018	..	2·5	2·95	5·0	3·8
VI.	10·068	,,	4·0	2·63	7·1	6·0
VII.	30·2	0·663	18·9	0·65	8·1	30·5
VIII.	50·32	0·656	35·4	0 516	7·3	55·2
IX.	100·68	0·627	84·7	0·141	7·6	124
X.	261·4	0·560	157·5	0·0614	5·4	274

The magnitude of *P*, compared with the change of the ratio of the halogens, in solution, is nearly constant from experiments V to IX, hence in the presence of both Cl′ and Br′ ions the conc of Ag′ ion is always definite in these experiments. This is only possible when the magnitude of R'_2 lies within certain limiting values the minimum of which is probably 0·004 (cf. Expt. VI, Table IV*a*) and the maximum is nearly unity. When R'_2 is beyond these limits, the conc. of Ag′ ion varies with one of the halogens in the sense of the solubility-product of the respective salt. The distribution of halogens in the precipitate and in the solution under such conditions is indefinite, hence the abnormal magnitude of π in experiments with low conc. of Br′ ion (cp. Table II).

When R'_2 is however within these limits, and the precipitate contains both AgCl and AgBr, both exerting their respective solution pressures, with a rise in the conc. of one of the halogen ions, say, Br′ ion the reaction will proceed in such a way that more of the Cl′ ion will strive to pass out from the precipitate to the solution. But AgBr in the precipitate will also exert its own solubility pressure and try to alter the concs. of Ag′ and Br′ ions according to its solubility product.

The two forces tend to counteract each other, an in the final equilibrium, the concs. of the three ions readjust themselves in such a way that the diminution of Br′ ion is compensated by a rise of Ag′ and Cl′ ions. In the experiments we are considering, the amount of total halogens in the precipitate is always about 1/100 that of the total halogens in the solution. Therefore, the transformation of one halogen salt in the precipitate to another will proceed in a definite way in the system, and the final composition of the mixed precipitate will depend upon the ratio of the halogens in the solution. That is, the ratio of $R_1 : R'_1$ will be constant.

Finally, the influence of solubility of the two sparingly soluble salts on the equilibrium will be considered

We have seen that in any experiment

$$\frac{AgBr}{AgCl} : \frac{Br'}{Cl'} = \pi',$$

which is nearly constant, i.e.

$$\frac{AgBr}{Br'} \times \frac{Cl'}{AgCl} \quad \text{Constant.}$$

Taking the conc of Ag′ ion (corresponding to any experiment) both in numerator and denominator, we get :

$$\frac{AgBr}{Ag' \times Br'} \times \frac{Ag' \times Cl}{AgCl} \quad \text{Constant.}$$

$$\text{Let } S_1 = \frac{Cl' \times Ag'}{AgCl} \text{ and } S = \frac{Ag' \times Br'}{AgBr}.$$

We have already seen that in Küster's experiments (III to IX) the ratio

AgBr : KBr, i e., AgBr : Br′ is equal to a constant quantity.

This however does not hold good in experiments with higher concentrations of Br′ ion. Thus, it will be of interest to note, it has a similar ratio, viz.

$$\frac{Ag' \times Br'}{AgBr}$$

in different experiments.

Again, since,

$$S_1 \times S_2 = \text{Constant},$$

if one of the factors in the left hand side be constant the other will also be constant. In the following table these quantities are recorded :—

Table IVB.

Concentrations are given in gram-mols per litre.

Exp. No. (Table III)	AgBr $\times 10^{-3}$	AgCl $\times 10^{-3}$	Cl′ ion	Br′ ion	Ag′ $\times 10^{-10}$	$S_1 \times 10^{-7}$	$S_2 \times 10^{-10}$	A.	B.	$\frac{B}{A}$.
III.	0·728	9·31	0·675	0·3	3·35	0 24	1·4	125	210	1·6
IV.	3 24	6·84	0 673	1·78	3·09	0 3	1·7	160	260	1·6
V.	4 5	5·58	,,	2·53	2·95	0·35	1·69	185	260	1·4
VI.	6·05	4·03	,,	4·0	2·63	0·44	1·73	220	265	1·2
VII.	9·16	0·92	0·663	18·9	2·65	0 46	1·36	250	210	0·84
VIII.	9·63	0·45	0 656	35·4	0·316	0 46	1·2	240	185	0·8
IX.	9·9	0·18	0 627	84 7	0 144	0·5	1·2	260	185	0·7
X.	10·0	0·096	0 560	157·5	0·0614	0·36	0·96	190	150	0·8

It will be noticed that both S_1 and S_2 are nearly constant in the last six experiments.

Where we have only pure AgCl or AgBr, the expressions S_1 and S_2 are of no significance. But in the experiments, we are considering there actually exists a rather remarkable relation between components in the solid phase and those in the liquid phase ; hence, the following discussion seems to be permissible.

The ratios of the absolute value of S_1 and S_2 to the solubility-products of AgCl and AgBr are represented in the table by A and B. It will be noticed that both A and B are nearly constant and also that the ratio A : B may roughly be assumed to be equal to *unity*. Though the absolute magnitudes of the former are of no importance, they however point out the following remarkable relation :

$$\pi' = \frac{B}{A} \times \frac{S_1}{S_2} = \frac{SP \text{ of AgCl}}{SP \text{ of AgBr}}$$

As a matter of fact, the ratio of the two solubility-products are :

$$\frac{1 \cdot 9 \times 10^{-10}}{6 \cdot 5 \times 10^{-13}} = 290$$

It will be seen in Table III, that the mean value of π, viz. 294 agrees remarkably with the above one in which the solubilities of AgCl and AgBr are taken from Theil.[1]

The magnitudes of the solubilities of the sparingly soluble salts, evidently, exert a significant influence upon the equilibrium which we are considering.

Conclusions.

I To explain the equilibrium in the fractional precipitation of AgCl and AgBr, it is not necessary to assume the existence of an Isomorphous mixture of the two salts. It can be treated as a general case.

II In fractional precipitation of two sparingly soluble salts, the more insoluble one exerts a greater influence, in the readjustment of the components in the solution and in the precipitate.

III In Küster's experiments (X to XVI table II) the concentration of the halogens in the solution and the small quantity of $AgNO_3$ used are quite suitable for the more sparingly soluble component viz. Br′ ion to manifest its influence in a definite and regular manner. Hence the ratio of Bromine to Chlorine in the precipitate alters in a definite way with regard to halogens in the solution.

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[1] *Zeit. anorg. Chem.*, *21*, (1900).

SOME METALLIC ARSENATES AND PHOSPHATES.

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The aim of this paper is a comparative study of some of the metallic arsenates and phosphates. The comparison between these rests mainly on their mode of formation and composition. An attempt has also been made to examine closely one of their interesting physical properties, that is the molecular volumes of some of the arsenates and their water of crystallisation and a striking result has been obtained in this connection.

As regards the arsenates, salts of eighteen different metals Co, Ni, Mn, Zn, Hg, Mg, Cu, Pb, Bi, Sn, Ag, Gl, Fe, Cr, U, Th, Ce and V have been studied. The arsenates are produced by two distinct methods. Firstly, by a double decomposition of a salt solution of one of the metals with a solution of a sodium arsenate $Na_2HAsO_4.xH_2O$, and secondly, of a salt solution with the powdered insoluble calcium arsenate $CaHAsO_4.xH\ O$, in suspension in water. The former takes place in the cold and the latter in the boiling state, the salt whose arsenate is to be formed being kept in great excess in both cases. The chief point to note is that by these two simple reactions instead of obtaining one and the same arsenate, products of different composition are often formed.

As regards the phosphates, a similar investigation was made. Phosphates of only six metals were studied, viz. Co, Ni, Mn, U, Th and Ce. They were produced by the action of different metallic salts with sodium phosphate $Na_2HPO_4.xH_2O$ and calcium phosphate $Ca_3(PO_4)_2.xH_2O$. For the first two metals, the phosphates obtained by both the methods have been described, whereas for the last four, only the second method has been tried. Nickel and cobalt gave by both the methods one and the same salt, viz. $Ni_3(PO_4)_2.7H_2O$ and $Co_3(PO_4)_2.8H_2O$ respectively. Mn, U, Th and Ce gave on the other hand, $Mn_3(PO_4)_2.5H_2O$, $UHPO_4.5H_2O$, $ThP_2O_7.6H_2O$ and $CePO_4.2H_2O$, respectively. It is to be noted here that the last three salts are exactly analogous to the corresponding arsenates $UHAsO_4.5H_2O$, $ThAs_2O_7.6H_2O$ and $CeAsO_4.2H_2O$, obtained in a similar manner.

The literature of arsenates and phosphates is a vast one and well-known authors like Berzelius, Rose, Graham, Coloriano, Mitscherlic, Wittstein, Salkowski, Lefevers, etc., have worked in the field for more than a century. References to the known compounds are given in the practical portion of this paper along with the new ones described therein. To sum up, the following arsenates and phosphates have been mentioned: $Co_3(AsO_4)_2.6H_2O$, $Co_3(AsO_4)_2.3H_2O$ and $Co_3(PO_4)_2.8H_2O$; $Ni_3(AsO_4)_2.8H_2O$ and $Ni_3(PO_4)_2.7H_2O$; $Zn_3(AsO_4)_2.3H_2O$ and $ZnHAsO_4.3H_2O$; $Cu_3(AsO_4)_2.6H_2O$; $Mg_3(AsO_4)_2.8H_2O$; $Cd_3(AsO_4)_2.H_2O$ and $CdHAsO_4$; $CeAsO_4.2H_2O$, $CeAsO_4.5H_2O$, and $CePO_4.2H_2O$; $Sn_3(AsO_4)_4.8H_2O$; $Fe_2(AsO_4)_2.xH_2O$, and $4Fe_2O_3.3As_2O_5.21H_2O$; $Ag_3(AsO_4)$; $Bi(AsO_4)$, and $Bi_4(As_2O_7)_3$; $Hg_3(AsO_4)_2$ and $Hg_3(AsO_3)_2$; $5MnO.2As_2O_5.5H_2O$, $MnHAsO_4.3\frac{1}{2}H_2O$ and $Mn_3(PO_4)_2.5H_2O$; $Cr_2(AsO_4)_2.10H_2O$ and $3Cr_2O_3.2As_2O_5.28H_2O$; $4GlO.As_2O_5.9H_2O$ and $4GlO.As_2O_5.7H_2O$; $5V_2O_5.As_2O_5.4H_2O$ and $2V_2O_5.As_2O_5.10H_2O$; $PbHAsO_4$ and $3PbO.2As_2O_5$. $UHAsO_4.5H_2O$, $(UO_2)HAsO_4.1\frac{1}{2}.H_2O$ and $UHPO_4.5H_2O$; $ThAs_2O_7.6H_2O$ and $ThP_2O_7.6H_2O$.

Having obtained normal arsenates of some of the metals with different amounts of water of crystallisation, e.g. $Co_3(AsO_4)_2.6H_2O$ and $Co_3(AsO_4)_2.3H_2O$, and remembering that molecular volume of water of crystallisation of arsenates of heavy metals have not been studied, it occurred to me that there might be some interesting relation existing in the different molecular volumes of the water of crystallisation of these arsenates. With this view in mind the specific gravities of such arsenates both in the hydrated and the dehydrated states were determined, whence the molecular volumes of each molecule of water of crystallisation in them were calculated as will be noticed later on.

EXPERIMENTAL.

The Arsenates.

$Co_3(AsO_4)_2.3H_2O$.

About 12 gms. of cobalt chloride in 200 c.c. of water were treated with 3 gms. of calcium arsenate $CaHAsO_4$ in suspension in 60 c.c. of water in the boiling state with constant stirring. A rapid reaction ensued and a very fine pink precipitate was formed. The whole mixture was kept boiling for half an hour and set aside overnight. It was washed next day several times by decantation with hot water and finally washed on the pump free of chloride, etc., and dried on a porous plate at the room temperature of 29°-30° for a day. The salt is insoluble in water but dissolves in dilute acids.

I. 0·2072 gave 0·0726 Co and 0·1262 $Mg_2As_2O_7$;—CoO=44·53, As_2O_5=45·13.

The arsenate obtained above was boiled once more in a concentrated solution of cobalt chloride for another ten minutes and washed and dried as before. The second crop on analysis proved to be of the same composition.

II. 0·3071 gave 0·1054 Co, and 0·1837 $Mg_2As_2O_7$ and 0·4961 gave 0·0569 H_2O; CoO=43·59, As_2O_5=44·28, H_2O=11·46, $Co_3(AsO_4)_2.3H_2O$ requires CoO=44·20, As_2O_5=45·18 H_2O=10·62.

$Co_3(AsO_4)_2.6H_2O$.

To about 15 gms. of cobalt chloride in 250 c.c. of water in the cold were added in a thin stream 10 gms. of Na_2HAsO_4 in 50 c.c. of water. The precipitate obtained was washed with cold water, and dried on a porous plate at the Temperature of 30°c. for two days.

I. 0·4051 gave 0·1280 Co, and 0·2206 $Mg_2As_2O_7$;—CoO—40·17, As_2O_5=40·29..

II. 0·3139 gave 0·0974 Co, and 0·7168 gave 0·1419 H_2O;—CoO=39·58, H_2O=19·74; $Co_3(AsO_4)_2.6H_2O$ requires CoO=39·96, As_2O_5=40·85 H_2O=19·19. *

$Ni_3(AsO_4)_2.8H_2O$.

A salt of the above composition was obtained by the methods described above. By the "$CaHAsO_4$"-method, 0·3705 gave 0·1360 NiO, and 0·1906 $Mg_2As_2O_7$;—NiO=36·70, As_2O_5=38·08.

By the "$NaHAsO_4$"-method, 0·3226 gave 0·1199 NiO and 0·1675 $Mg_2As_2O_7$.NiO=37·17, As_2O_5=38·43; $Ni_3(AsO_4)_2.8H_2O$ requires CoO=37·45, As_2O_5=38·45 H_2O=24·10.†

$5MnO.2As_2O_5.5H_2O$.

20-25 gms. of manganese chloride were dissolved in 150 c.c. of water to which 5 gms. of calcium arsenate in 100 c.c. of water were added slowly in the boiling state. The subsequent details were as noted in

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* $Co_3(AsO_4)_2.8H_2O$; Karsten (*Pogg. Ann.* 60, 266).

† $Ni_3(AsO_4)_2.2H_2O$; Coloriano (*Bull. Soc. Chem.* 45, 24).

the preceding. The salt was dried at 27·5°C. for a day. It has very faint pink colour and dissolves in dilute acids. The probable reactions are :—

$CoHAsO_4 + MnCl_2 = MnHAsO_4 + CoCl_2$, $5MnHAsO_4 + 5H_2O = 5MnO.2As_2O_5.5H_2O + H_3AsO_4 + H_2O$.

I. 0·400 gave 0·3219 $Mn_2P_2O_7$ and 0·2767 As_2O_5 ;—MnO=40·20; As_2O_5=51·25.

II. 0·2357 gave 0·1920 $Mn_2P_2O_7$ and 0·1575 As_2S ;—MnO=40·65, As_2O_6=49·51.

A second crop was produced by boiling the first with a concentrated solution of manganese chloride. The composition remained constant as was the case with $Co_3(AsO_4)_2.3H_2O$.

0·3456 gave 0·2758 $Mn_2P_2O_7$ and 0·2324 As_2S_5 ;—MnO=39·87; As_2O_5 =49·82. $5MnO.2As_2O_5 . 5H_2O$ requires MnO=39·22 ; As_2O_5=50·82.

Coloriano obtained the above salt by the action of Na_2HAsO_4 on a manganese salt in solution and by boiling the salt formed in water.

$MnHAsO_4.3\frac{1}{2}H_2O$.

This arsenate was produced by the interaction of a solution of Na_2HAsO_4 and $MnCl_2$ (7 : 20) in the cold.

I. 0·4215 gave 0·2470 $Mn_2P_2O_7$ and 0·2446 As_2S_5 ;—Mn=22·67 ; AsO_4=51·98.

II. 0·5191 gave 0·3078 $Mn_2P_2O_7$ and 0·3251 As_2S_5 ;—Mn=21.67 ; AsO_4=53·02. Calculated for $MnHAsO_4.3\frac{1}{2}H_2O$: Mn=21·31 ; AsO_4= 53·8.

$Zn_3(AsO_4)_2.3H_2O$.

About 24 gms. of zinc chloride were dissolved in 150 c.c. of water just acidified with hydrochloric acid. To this solution were added 6 gms. of calcium arsenate in the usual manner.

I. 0·3556 gave 0·3610 $Zn(NH_4)PO_4$ and 0·2025 As_2S_5 ;—ZnO= 46·29, As_2O_5=42·21.

II. 0·2815 gave 0·2880 $Zn(NH_4)PO_4$ and 0·1620 As_2S_5 and 0·3261 gave 0·0336 H_2O ;—ZnO=46·65 ; As_2O_5=42·64 ; H_2O=10·30. $Zn_3(AsO_4)_2.3H_2O$ requirers ZnO=46·21 ; As_2O_5=43·56 ; H_2O=10·23.

$ZnHAsO_4.3H_2O$.

A salt of the above composition was produced by bringing together a solution of sodium hydrogen arsenate and zinc chloride (8 : 20) in the cold.

I. 0·2131 gave 0·1470 $Zn(NH_4)PO_4$ and 0·1283 As_2S_5;—Zn=25·26; AsO_4=53·9.

II. 0·2883 gave 0·2003 $Zn(NH_4)PO_4$ and 0·1734 As_2S_5;—Zn=25·44; AsO_4=53·86. $ZnHAsO_4.3H_2O$ requires Zn=25·19; AsO_4=53·66H_2O=21·24.*

$4Fe_2O_3.3As_2O_5.21H_2O$.

This arsenate was produced by the action of calcium arsenate on a solution of ferric chloride, the proportions being 1 : 6. The other details were exactly as described before. It was tried on a porous plate at 23°-25° C. for four days.

I. 0·2239 gave 0·0815 Fe_2O_3 and 0·1210 As_2S_5;—Fe_2O_3=36·41; As_2O_5=40·05.

II. 0·3699 gave 0·1348 Fe_2O_3; –Fe_2O_3=36·44. $4Fe_2O_3.3As_2O_5.21H_2O$ requires Fe_2O_3=37·52; As_2O_5=40·39; H_2O=22·09.

$Fe_2(AsO_4)_2.xH_2O$.

A solution of ferric chloride containing 24 gms. of the salt in 15 c.c. of water was treated in the cold with a solution of 10 gms. of sodium hydrogen arsenate in 100 c.c. of water. A gelatinous creamy white mass was obtained which dissolved in excess of ferric chloride solution. By adding the whole of the arsenate solution to the ferric salt a product was finally produced. It was set aside overnight, after which it was washed and dried for five days at 24°—25° C.

Under ordinary atmospheric conditions it was found to be extremely efflorescent. A salt of constant weight was however obtained by keeping the same over concentrated sulphuric acid in a dessicator for a long time.

0·4283 gave 0·1772 Fe_2O_3 and 0·3393 As_2S_5;—Fe_2O_3=41·38; As_2O_5=58·70. $Fe_2(AsO_4)_2$ requires Fe_2O_3=41·03; As_2O_5=58·97.†

$3Cr_2O_3.2As_2O_5.28H_2O$.

This was produced by the action of calcium arsenate on a solution of chromic chloride, in the usual manner.

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* An arsenate $Zn_3(AsO_4)_2.3H_2O$ is described by Kottig (*J. Prakt. Chem.* 48, 182). Debray also mentions of $ZnHAsO_4.H_2O$ (*Bull. Soc. Chem.* (2), 2, 14).

† An acid arsenate $2Fe_2(HAsO_4)_3.9H_2O$ obtained as a white precipitate by adding Na_2HAsO_4 to a solution of ferric chloride is described by Roscoe (*Treatise of Chemistry* 1913] Vol. II, 1246.)

The salt has a fine deep green colour and dissolves in moderately strong hydrochloric acid.

I. 0·2603 gave 0·0832 Cr_2O_3 and 0·1123 As_2S_5;—Cr_2O_3=31·96; As_2O_5=31·96.

II. 0·2593 gave 0·0832 Cr_2O_3 and 0·1154 As_2S_5;—Cr_2O_3=32·08; As_2O_5=32·98.

$3Cr_2O_3 . 2As_2O_5.28H_2O$ requires Cr_2O_3=32·34; As_2O_5 = 32·68; H_2O_5 =35·08.

$Cr_2(AsO_4)_2.10H_2O$.

This arsenate was obtained by bringing together a concentrated solution of chromic chloride with a solution of sodium hydrogen arsenate.

I. 0·3828 gave 0·2117 As_2S_5; and 0·1043 Cr_2O_3;—As_2O_5=40·98; Cr_2O_3=27·24.

II. 0·2867 gave 0·1562 As_2S_5 and 0·0770 Cr_2O_3;—As_2O_5=40·36; Cr_2O_3=26·85.

$Cr_2(AsO_4)_2.10H_2O$ requires As_2O_5=40·92; Cr_2O_3=27·04; H_2O= 32·04.

During the estimation of water of hydration of the salt it was noticed that the substance began to give off water from 105° C. the dehydration was perfomed at 170°C. and it was found that at this temperature the salt lost only 8 out of 10 mols. of water of crystallisation.

I. 0·3950 gave 0·0965 H_2O;—H_2O=24·43.

II. 0·3661 gave 0·0890 H_2O;—H_2O=24·31.

$Cr_2(AsO_4)_2.8H_2O$ requires H_2O=24·51.*

$4BeO.As_2O_5.9H_2O$.

About 4 gms. of Beryllium oxide were just dissolved in dilute hydrochloric acid and the barely acid solution made up to 150 c.c. This was treated in the boiling state with about 3 gms. of $CaHAsO_4$ in the usual manner. A fine white precipitate was obtained, which was washed and dried

I. 0·3231 gave 0·0649 BeO and 0·2008 As_2S_5;—BeO=20·05; As_2O_5=45·99.

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* No normal arsenate of the type $Cr_2(AsO_4)_2.xH_2O$ is known. The corresponding phosphate $Cr_2(PO_4)_2.12H_2O$ however is mentioned by Rammelsberg (*Pogg. Ann. 68*, 383); Etard (*C. R.*, *84*, 1091).

II. 0·5318 gave 0·1052 GlO and 0·3338 As_2S_5 ;—BeO=19·78; As_2O_5 =46·50.

$4BeO.As_2O_5.9H_2O$ requires BeO=20·32; As_2O_5=46·74.

$4BeO.As_2O_5.7H_2O$.

This salt was obtained by treating 3 gms. of Beryllium oxide in hydrochloric acid solution and 3 gms. of Na_2HAsO_4 solution in the cold.

0·2861 gave 0·0599 BeO and 0·1907 As_2S_5 ;—BeO=20·93 ; As_2O_5= 51·71.

$4BeO.As_2O_5.7H_2O$ requires BeO=21·92; As_2O_5=50·43.

$Cu_3(AsO_4)_2.6H_2O$.

28 grms. of cupric chloride in 300 c.c. of water was treated in the boiling state with 5 gms. of $CaHAsO_4$. The product was dried for two days at 22°–23° C. The salt is blue.

0·4262 gave 0·2272 $Mg_2As_2O_7$ and 0·1764 Cu_2S ;—As_2O_5=39·47; CuO=41·37.

$Cu_3(AsO_4)_2.6H_2O$ requires As_2O_5=39·93;—CuO=41·31 ; H_2O= 18·76.

The salt gave up water from 105° C. onwards. When the drying was performed at 180° C. it gave up water corresponding to $3H_2O$ and not to $6H_2O$. When submitted to a higher temperature of 210° C. no further setting free of water of hydration was noticed.

0·3117 gave 0·0340 H_2O ; —H_2O=10·91.

$Cu_3(AsO_4)_2.3H_2O$ requires H_2O=9·38.

By treating a solution of cupric chloride with Na_2HAsO_4 solution (8 : 5), an arsenate of the same composition was obtained. This also gave up only $3H_2O$ out of it $6H_2O$.

0·3600 gave 0·1922 $Mg_2As_2O_7$ and 0·1478 Cu_2S and 0·4397 gave 0·0483 H_2O at 180° C ;—As_2O_5=39·52; CuO=41·04; H_2O=10·98.*

$Ag_3(AsO_4)$.

Silver arsenate of the above composition was produced by both the "$CaHAsO_4$" and the "Na_2HAsO_4"-method. It has a chocolate colour. †

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* Coloriano mentions of the arsenate $Cu_3(AsO_4)_2.4H_2O$ (*C.R.*, *103*, 273).

† Joly (*C.R.*, *103*, 1071) obtained the same salt by the addition of arsenic acid or an alkali arsenate to a silver nitrate solution.

$Hg_3(AsO_4)_2$.

About 20 gms. of mercuric nitrate were just dissolved in nitric acid and the solution made up to 200 c.c. This was made to react with 5 gms. of $CaHAsO_4$ in the boiling state. A heavy pale yellow precipitate was formed which was washed and dried as usual.

I. 0·3618 gave 0·2869 HgS;—Hg=68·37.

II. 0·4385 gave 0·3460 HgS; – Hg=68·01.

$Hg_3(AsO_4)_2$ requires Hg=68·37.*

$Hg(AsO_3)_2$.

This was produced by bringing together a solutian of 20 gms. of $HgNO_3$ in nitric acid with a solution of Na_2HAsO_4 in the cold. The heavy yellowish white precipitate was kept aside for five hours, and finally washed and dried as usual.

The salt contained no water of crystallisation and proved to be stable even at 180° C.

Simon (*Pogg. Ann.*, *41*, 424) prepared the corresponding mercurous salt $Hg_2(AsO_3)_2$ in the following manner. He first obtained the salt Hg_2HAsO_4 by the interaction of an excess of arsenic acid and mercurous nitrate. By adding an excess of H_3AsO_4 to the salt $Hg_2H(AsO_3)_2$ and on evaporating the mixture $Hg_2(AsO_3)_2$ was produced as a white powder.

The formation the salt can be explained thus—

$2HgNO_3 + H_3AsO_4 = Hg_2HAsO_4 + 2HNO_3$.

$Hg_2HAsO_4 + H_3AsO_4 = Hg_2\ (AsO_3)_2 + 2H_2O$.)

A similar explanation might be offered for the formation of $Hg(AsO_3)_2$.

$Hg(NO_3)_2 + Na_2HAsO_4 = HgHAsO_4 + 2NaNO_3$.

$HgHAsO_4 + Na_2HAsO_4 + 2HNO_3 = Hg(AsO_3)_2 + 2NaNO_3 + 2H_2O$.

$3PbO.2As_2O_5$.

This was obtained by the action of $CaHAsO_4$ on a just acid solution of lead nitrate (4 : 10).

I. 0·4960 gave 0·4011 $PbSO_4$;—PbO=59·50.

II. 0·3810 gave 0·3061 $PbSO_4$;—PbO=59·12.

$3PbO.2As_2O_5$ requires PbO = 59·25.

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* The mercurous salt $(Hg_2)_3(AsO_4)_2$ exists Coloriano (*C R.*, *103*, 273). The corresponding phosphate $Hg_3(PO_4)_2$ is said to be obtained by adding Na_2HPO_4 to a slightly acid solution of $Hg(NO_3)$ (Haack, *Chemisches Centralblatt*, *2*, 736 [1890]).

$PbHAsO_4$.

This acid arsenate was the result of the interaction Na_2HAsO_4 with a barely acid solution of lead nitrate (8 : 15).

0·3270 gave 0·2874 $PbSO_4$, and 0·1474 As_2S_5 ;—PbO=64·69; As_2O_5=33·47.

$PbHAsO_4$ requires PbO=64·29; As_2O_5=33·13.

$BiAsO_4$.

The above bismuth salt was obtained by bringing together a just acid solution of bismuth nitrate and $CaHAsO_4$ in the usual manner.

I. 0·6601 gave 0·4950 Bi_2S_3 ;—Bi=60·92.

II. 0·3618 gave 0·2699 Bi_2S_3 ;—Bi=60·49.

$BiAsO_4$ requires Bi=59·95.

Scheider [*J. prakt. Chem.*, (2) *20*, 418] and Salkowsky [Ibid, *104*, 129] obtained $BiAsO_4.\frac{1}{2}H_2O$ by the interaction of arsenic acid on an acid solution of bismuth nitrate.

Compound—$Bi_4(As_2O_7)_3$.

This was produced by treating a solution of bismuth nitrate and Na_2HAsO_4 in presence of nitric acid (8: 7).

I. 0·2853 gave 0·1950 Bi_2S_3 ;—Bi=55·54.

II. 0·3264 gave 0·2201 Bi_2S_3 ;—Bi=54·80.

$Bi_4(As_2O_7)_3$ requires Bi=55·66.

$Cd_3(AsO_4)_2.H_2O$.

This arsenate was the result of the interaction $CaHAsO_4$ with cadmium chloride (5: 20) in the usual way.

0·5196 gave 0·3163 CdO, and 0·2027 As_2S_3 ;—CdO=60·87; As_2O_5=36·44.

1·0480 gave 0·0246 H_2O at 190° C ;—H_2O=2·46. $Cd_3(AsO_4)_2.H_2O$ requires CdO=60·75; As_2O_5=36·39 H_2O=2·86.

$CdHAsO_4$.

The above salt was obtained by treating a solution of cadium chloride with Na_2HAsO_4.

0·5954 gave 0·3060 CdO, and 0·2931 As_2S_3 ;—Cd=44·99; AsO_4= 55·58.

$CdHAsO_4$ requires Cd=44·62; AsO_4=55·38.

Salkowaski obtained the salt $Cd_3(AsO_4)_2 . 3H_2O$ by the interaction of Na_2HAsO_4 on $CdCl_2$ solution and not the salt $CdHAsO_4$. It is probable that the reactions proceed thus $Na_2HAsO_4+CdCl_2=CdHAsO_4+2NaCl.3CdHAsO_4 \longrightarrow Cd_3(AsO_4)_2+3H_3AsO_4$.

$Mg_3(AsO_4)_2.8H_2O$.

This was obtained by the interaction Na_2HAsO_4 with magnesium chloride, the proportions being 6 : 10.

0·3982 gave 0·2691 $Mg_2P_2O_7$ and 0·2497 As_2S_5.

0·7081 gave 0·2032 H_2O at 192°C.;—$MgO=24·47$; $As_2O_5=46·46$; $H_2O=28·69$.

$Mg_3(AsO_4)_2. 8H_2O$ requires $MgO=24·44$; $As_2O_5=46·46$; $H_2O=29$.

$Sn_3(AsO_4)_4.8H_2O$.

The above stannic arsenate was obtained by bringing together a solution of stannous chloride containing 15 gms. of the salt and 5 gms. of $CaHAsO_4$ in the boiling state.

I. 0·3388 gave 0·1946 SnO_2; and 0·1508 As_2S_5;—$SnO_2=57·44$; $As_2O_5=32·98$.

II. 0·318 gave 0·1824 SnO_2 and 0 1408 As_2S_5.
0.6933 gave 0·0674 H_2O at 180° C.;—$SnO_2=57·33$; $As_2O_5=32·79$; $H_2O=972$.

$Sn_3(AsO_4)_4.8H_2O$ requires $SnO_2=57·28$; $As_2O_5=32$ $53H_2O=10·19$. Cf. $Sn_3(AsO_4)_4.6H_2O$; Williams (*Proc. Chem. Soc.*, Manchester, *15*, 67).

$ThAs_2O_7.6H_2O$.

8 gms. of thorium nitrate were dissolved in 150 c.c. of water. To this were added 3 gms. $CaHAsO_4$ in the usual way.

The formation of the salt can be represented thus :—

$Th(NO_3)_4+CaHAsO_4=Th(HAsO_4)_2+2Ca(NO_3)_2$.

$Th(HAsO_4)_2 \longrightarrow ThAs_2O_7+H_2O$.

I. 0·2542 gave 0·1105 ThO_2 and 0 1306 $Mg_2As_2O_7$;—$ThO_2=43·37$; $As_2O_5=38·03$.

II. 0·3350 gave 0·1450 ThO_2 and 0·1720 As_2S_5;—$ThO_2=43·29$; $As_2O_5=38·05$.

$ThAs_2O_7. 6H_2O$ requires $ThO_2=43·85$; $As_2O_5=38$ 25.

No pyroarsenate of thorium seems to exist. Cleve mentions of the corresponding phosphate $ThP_2O_7.2H_2O$.

$UHAsO_4.5H_2O$.

This salt was produced by treating a solution of uranium acetate with calcium hydrogen arsenate in the boiling state (6 : 3).

I. 0·4004 gave 0·1308 As_2S_5 and 0·2385 U_3O_8 ;—AsO_3=29·85; U=50·53.

II. 0·3850 gave 0·1266 As_2S_5 and 0·2301 U_3O_8 ;—AsO_4=29·45; U=50·69.

$UHAsO_4.5H_2O$ requires AsO_4=29·72; U=50·85.

$(UO_2)HAsO_4.1\frac{1}{2}H_2O$.

The above salt was obtained by adding a solution of Na_2HAsO_4 to a solution of uranium acetate. It is slightly soluble in water.

I. 0·2741 gave 0·1675 UO_2 and 0·1011 As_2S_5 ;—UO_2=60·96; AsO_4=33·04.

II. 0·2116 gave 0·1283 UO_2 ;—UO_2=60·43.

$(UO_2)HAsO_4.1\frac{1}{2}H_2O$ requires UO_2=60·58; AsO_4=33·42.

Ebelmen describes $(UO_2)HAsO_4.4H_2O$ [*Ann. chem. Phys.*, (3), *5*, 220].

$5V_2O_5.As_2O_5.4H_2O$.

A solution of vanadium chloride containing 6 gms. of the salt in 150 c.c. of water was treated with 3 gms. of $CaHAsO_4$ in the boiling state. A rapid reaction took place and a pale violet flocculent precipitate was at first obtained which on prolonged boiling assumed a fine granular structure. The final product dried on a porous plate was of green colour. This was once more treated with a boiling solution of fairly concentrated vanadium chloride solution and washed and dried as usual.

I. 0·3031 gave 0·0626 As_2S_3 and 12·4 c.c. N/5 $KMnO_4$ were used up in the titration of the vanadium :—As_2O_5=19·30; V_2O_5=74·09.

II. 0·1770 gave 0·0326 As_2S_3 and 28·82 c.c. $N/_{20}$ $KMnO_4$ were used up in titrating the vanadium ;—As_2O_5=19·32; V_2O_5=75·02. $5V_2O_5.As_2O_5.5H_2O$ requires As_2O_5=18·97; V_2O_5=75·08; H_2O=5·95.

$2V_2O_5.As_2O_5.10H_2O$.

This compound was obtained by the interaction of vanadium chloride with sodium arsenate in the cold.

The final product was green and was found to be very slightly soluble in water.

I. 0·2024 gave 0·0629 As_2S_3 and 20·92 c.c. $N/_{20}$ $KMnO_4$ were used up in the titration of the vanadium;—As_2O_5=29·03; V_2O_5=47·13.

II. 0·4712 gave 0·1458 As_2S_3 and 48·8 c.c. of $N/_{20}$ $KMnO_4$ were used up in the titration of the vanadium;—As_2O_5=28·89; V_2O_5=47·21.

$2V_2O_5.As_2O_5$ $10H_2O$ requires As_2O_5=29·71; V_2O_5=47·02; H_2O=23·29.

The compound $As_2O_5.V_2O_5.10H_2O$ also called arsenovanadic acid is said to be obtained when V_2O_5 is heated with arsenic acid. It forms yellow crystals.

It therefore appears that the above two compounds are quite different products.

$CeAsO_4.2H_2O$.

This salt was produced by treating 5 gms. of cerium nitrate with 2 gms. of $CaHAsO_4$ in the usual manner.

I. 0·3339 gave 0·1808 CeO_2 and 0·1625 $Mg_2As_2O_7$;—Ce=43·98; AsO_4=43·91.

II. 0·2652 gave 0·1435 CeO_2 and 0·1304 As_2S_5;—Ce=44·04; AsO_4 =44·03.

$CeAsO_4.2H_2O$ requires Ce=44·30; AsO_4=43·99; H_2O=11·71.

$CeAsO_4.5H_2O$.

This arsenate was obtained by adding a solution of Na_2HAsO_4 to cerium nitrate in the cold (5 : 3).

I. 0·2981 gave 0·1375 CeO_2 and 0·1255 $Mg_2As_2O_7$;—Ce=37·58; AsO_4=37·67.

II. 0·3854 gave 0·1792 CeO_2 and 0·1625 As_2S_5;—Ce=37·84; AsO_4 =37·67.

$CeAsO_4.5H_2O$ requires Ce=37·94; AsO_4=36·66; H_2O=24·40.

Recent literature points to ceric salts $Ce(H_2AsO_4)_2$ $4H_2O$, $Ce(HAsO_4)_2 6H_2O$. Barbieri and Calzolari (*Ber.*, 1910, *43*, 2214). The corresponding phosphate $CePO_4$ $2H_2O$ is however mentioned by Jolin and Harley (*T.C.S.*, 1882, *23*, 4120).

EXPERIMENTAL.

The Phosphates:—

$Co_3(PO_4)_2$. $8H_2O$.

The above normal cobalt phosphate was obtained by treating a solution of cobalt chloride containing 16 gms. of the salt in 250 c.c. of water with 5 gms. of calcium phosphate in the boiling state.

0·423 gave 0·1923 Co; and 0·1917 $Mg_2P_2O_7$;—CoO=45·50; P_2O_5=28·66.

$Co_3(PO_4)_2.8H_2O$ requires CoO=45·63; P_2O_5=28·80; H_2O=25·57.

A salt of the same composition was also produced by the interaction of Na_2HPO_4 with cobalt chloride solution.

0·4580 gave 0·2034 Co; 0·2058 $Mg_2P_2O_7$;—CoO=45·50; P_2O_5=28·66.

When this salt was dehydrated at 180° C. it gave up 6 out of its 8 molecules of water.

0·4703 gave 0·0943 H_2O;—H_2O=19·8.

That the salt was dehydrated so far was further verified by analysis.

0·3681 gave 0·1622 CoO and 0·2071 $Mg_2P_2O_7$;—CoO=56·03; P_2O_5=35·84.

$Co_3(PO_4)_2$. $2H_2O$ requires CoO=55·83; P_2O_5=35·25; H_2O=8·92.[*]

$Ni_3(PO_4)_2.7H_2O$.

An arsenate of the above composition was obtained by both the methods described in this paper.[†]

$Mn_3(PO_4)_2.5H_2O$.

This salt was produced by bringing together a solution of manganese chloride and calcium phosphate (12 : 8).

I. 0·3391 gave 0·1680 $Mg_2P_2O_7$ and 0·3229 MnP_2O_7;—P_2O_5=31·61; MnO=47·58.

II. 0·3801 gave 0·1898 $Mg_2P_2O_7$;—P_2O_5=31·85.

$Mn_3(PO_4)_2.5H_2O$ requires MnO=47·87; P_2O_5=31·91; H_2O=20·22.

Normal orthophosphates of the type $Mn_3(PO_4)_2.xH_2O$ are said to exist with 14, 9, 6 and 3 molecules of water of crystallisation.[‡]

$UHPO_4.5H_2O$.

This was obtained in a manner similar to the preparation of the salt $UHAsO_4$. $5H_2O$.

[*] $Co_3(PO_4)_2$. $8H_2O$. Regnose (*C.R.*, *34*, 795); Debray [*Ann chem. Phys.*, (3) *61*, 438].

[†] Rammelsberg. (*Pogg. Ann.*, *63*, 383).

[‡] Friend, *Inorganic Chemistry*, VIII [1915].

0·5103 gave 0·3368 U_3O_8 and 0·1315 $Mg_2P_2O_7$;—U=55·97 ; PO_4= 21·99.

$UHPO_4.5H_2O$ requires U=56·23 ; PO_4=22·35 ; H_2O=21·42. Cf.$UHPO_4.H_2O$ Rammelsberg (*Pogg. Ann.*, *59*, 1).

$ThP_2O_7.6H_2O$.

This pyrophosphate was formed exactly like $ThAs_2O_7.6H_2O$.

0·4888 gave 0·2506 ThO_2 and 0·2091 $Mg_2P_2O_7$;—ThO_2=51·36 ; P_2O_5=27·63 ; $ThP_2O_7.6H_2O$ requires ThO_2=51·36 ; P_2O_5=27·63, H_2O=21·01.

$CePO_4.2H_2O$.

The mode of formation was like that of the arsenate, $CeAsO_4.2H_2O$.

0·4489 gave 0·2837 CeO_2 and 0·1857 $Mg_2P_2O_7$;—Ce=51·45 ; PO_4= 35·30 ; $CePO_4.2H_2O$ requires Ce=51·66 ; PO_4=35·06 ; H_2O=13·28. Hartley (*T.C.S.* *41*, 202, 1882) is said to have obtained the same salt by double decomposition between a cerous salt and an alkali phosphate or phosphoric acid.

Molecular Volumes of Arsenates and their water of Crystallisation.

Determination of densities. The Densities of the arsenates with respect to water have been determined at the laboratory temperatures in toluene with a specific gravity bottle. The air in the interstices of the salts was removed by placing the sp. gravity bottle together with the salt and toluene in a vacuum. In the following table the densities and molecular volumes of the hydrated salts are given :—

Table I.

	Salts.	Density.	Mol Vols. $=\frac{\text{Mol. Wt.}}{\text{Density.}}$
1.	$Co_3(AsO_4)_2.3H_2O$	4·262	119·40
2.	$Co_3(AsO_4)_2.6H_2O$	3·116	180·68
3.	$Ni_3(AsO_4)_2.8H_2O$	2·879	214·67
4.	$Zn_3(AsO_4)_2.3H_2O$	5·260	100·19
5.	$Cr_2(AsO_4)_2.10H_2O$	3·294	170·60
6.	$Cu_3(AsO_4)_2.6H_2O$	3·861	146·82
7.	$Cd_3(AsO_4)_2.H_2O$	4·099	154·42
8.	$Mg_3(AsO_4)_2.8H_2O$	1·711	289·30
9.	$Sn_3(AsO_4)_4.8H_2O$	3·442	307·08

The above salts were dehydrated and the densities determined similarly, whence their molecular volumes were ascertained. It is however to be pointed out that temperature conditions of the experiments permitted only the removal of eight and three molecules of water from $Cr_2(AsO_4)_2.10H_2O$ and $Cu_3(AsO_4)_2.6H_2O$ respectively.

Table II.

	Salts	Density.	Mol. Volumes.
1.	$Co_3(AsO_4)_2$	4·986	91·25
2.	$Co_3(AsO_4)_2$	4·844	93·93
3.	$Ni_3(AsO_4)_2$	5·687	79·83
4.	$Zn_3(AsO_4)_2$	6·683	70·80
5.	$Cr_2(AsO_4)_2.2H_2O$	6·770	56·41
6.	$Cu_3(AsO_4)_2.3H_2O$	3·916	119·50
7.	$Cd_3(AsO_4)_2$.	4·237	145·14
8.	$Mg_3(AsO_4)_2$	2·231	157·32
9.	$Sn_3(AsO_4)_2$.	4·413	206·88

Subtracting the molecular volume of the dehydrated salt from the corresponding hydrated salt and dividing the result by the number of water molecules in the same the molecular volume of one molecule of water of crystallisation of the salt is obtained. The following table is given for illustration :--

Table III.

	Salts.	M.V. of hydrated salts.	M.V. of dehydrated salts.	M.V. of water of crystallisation.
1.	$Co_3(AsO_4)_2·3H_2O$	119·40	91·25	9·38
2.	$Co_3(AsO_4)_2\ 6H_2O$	180·68	93·93	14·46
3.	$Ni_3(AsO_4)_2\ 8H_2O$	214·69	79·83	16·85
4.	$Zn_3(AsO_4)_2·3H_2O$	100·19	70·80	9·79
5.	$Cr_2(AsO_4)_2\ 8H_2O$	170·60	56·41	14·27
6.	$Cu_3(AsO_4)_2·3H_2O$	146·82	119·50	9·10
7.	$Cd_3(AsO_4)_2·H_2O$	151·42	145·14	9·28
8.	$Mg_3(AsO_4)_2·8H_2O$	289·30	157·32	16·49
9.	$Sn_3(AsO_4)_4·8H_2O$	307·08	206·88	12·52

Remarks.

The above table exhibits many points of interest. It is remakable that the molecular volume of the water of crystallisation of the salts $Co_3(AsO_4)_2.6H_2O$, $Cr_2(AsO_4)_2.8H_2O$, $Ni_3(AsO_4)_2.8H_2O$ and $Mg_3(AsO_4)_2.8H_2O$ are almost the same and vary within two units. The stannic arsenate $Sn_3(AsO_4)_2.8H_2O$ however has for the molecular volume of the water of hydration the value 12·52, and thus does not agree with the foregoing. On the contrary, it is striking indeed that the salts $Co_3(AsO_4)_2.3H_2O$, $Zn_3(AsO_4)_2.3H_2O$ and $Cn_3(AsO_4)_2\ 3H_2O$,—all possessing three molecules of water should have for their molecular volume of the water of hydration the values 9·38, 9·79, and 9·10 respectively—differing from one another within a few tenths of a unit. One should not overlook the difference between the molecular volume of the water of hydrations of the salts $Co_3(AsO_4)_2.6H_2O$ and $Co_3(AsO_4)_2.3H_2O$. It has been said that the property of molecular volume of the water of crystallisation is additive. Although 14·46 and 9·36 are not as 2 : 1. It is to be noted that molecular volume of the water of crystallization of the salt $Cd_3(AsO_4)_2.H_2O$ is 9·28.

In this connection the work of Kopp on the molecular volumes should be mentioned. He makes the general conclusion that in the substances containing only a small number of water molecules (1—3) the molecular volume of the water of crystallization is 12·4; in others containing a larger number (2–7) it is 13·4, whereas a third class containing the largest number its mean value is 15·3.

If Kopp's generalisation is accepted we can account for the equal molecular volume of water of crystallisation of $Cd_3(AsO_4)_2.H_2O$; $Co_3(AsO_4)_2.3H_2O$, $Zn_3(AsO_4)_2\ 3H_2O$ and $Cu_3(AsO_4)_2\ 3H_2O$—the first of these is monohydrated whereas the rest are trihydrated. Also we have the general value for the molecular volume of the water of hydration for the arsenates described above a mean value of 9 and a mean value of 15·5 for those containing (6–8) molecules of water of hydration.

Attention must also be drawn to the work of Thorpe* on certain sulphates. According to him the difference between the molecular volumes of the monohydrate salt and the dihydrate salt is 13·3, between the dihydrate and the trihydrate 14·5, between the trihydrate and the tetrahydrate 15·4 and between the hexahydrate and the heptahydrate 16·2. The first molecule of water which he calls the constitutional water occupies a volume equal to 10·7. These observations of Thorpe are in complete harmony with those recorded by Kopp. On the contrary the values for the molecular volumes of the

* *T.*, 1880, 37, 102.

water of crystallisation of nitrates nitrites and hysonitrites of the alkali and alkaline earth metals by Ray and De* do not agree with the results obtained by Thorpe and Kopp. Thus they obtain for the hyponitrites of Ca, Sr and Ba, the molecular volume of the water of crystallisation respectively as 13 56; 10 55 and 11·88.

The results of Thorpe go to show that the molecular volume changes from hydrate to hydrate and that it is more or less an additive property.†

Subsequent researches have indicated that Kopp's conclusions regarding the molecular volumes must be slightly modified. Thus Schiff many years ago showed that the members of certain hydrated salts have practically the same molecular volume. He has also shown that all the alums have a molecular volume of about 277, double sulphates of the form $M_2M''(SO_4)_2.6H_4O$ have a common molecular volume of 207 and all the vitriols, i.e. salts of the form $M''SO_4.7H_2O$ whether isomorphous or not, have the molecular volume 146.

From all that has been said above it is very difficult to draw a hard and fast rule regarding the molecular volume of the water of crystallisation of salts. Taking into consideration all the works carried on in this branch of investigation one can say this much that a distinct general rule holds when the molecular volumes of the water of crystallisation of the same class of salts is taken into consideration. Thus a particular regularity is noticeable in the molecular volumes of the water of hydration in the sulphates, nitrates, double sulphates, carbonates, etc. In what little has been said about the molecular volumes of the water of crystallisation of the arsenates a similar generalisation is maintained.

The study of molecular volumes of salts in Inorganic Chemistry has been up to date very scanty. A wider knowledge is called for to clear up various anomalies and to establish a more thorough generalisation. The little that has been achieved with regard to the molecular volumes of the arsenates in this paper speaks in favour of the results of the previous workers. Future investigation might throw more light.

In conclusion I beg to acknowledge my best thanks to Sir P. C. Ray, Dr. P. C. Mitter, and Rev. Father J. van Neste, for occasional help and encouragement during the course of this work.

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* *T*, 1908, *93*, 997, 1909, *95*, 66; and 1916, *109*.

† *T.*, *37*, 102, 1880.

ON THE MÖBIUS SURFACE AND CONE OF THE FOURTH DEGREE.

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The Möbius Surface and Cone of the fourth degree are the simplest [pitch $\frac{m}{2n}=1$] of one of the two types of Möbius Surfaces and Cones.[*] At the suggestion of my Professor, Dr. C. E. Cullis, who has investigated the properties of the Surface and Cone of the third degree,[†] the simplest of the other type, I have attempted to investigate the properties of the Surface and Cone of the fourth degree.

1. MÖBIUS SURFACE OF THE FOURTH DEGREE.

It is the skew surface or scroll generated by straight lines whose equations in terms of a parametric angle θ are

$$\frac{x-a\cos\theta}{\cos^2\theta}=\frac{y-a\sin\theta}{\sin\theta\cos\theta}=\frac{z}{\sin\theta}$$

It is the surface of type m even, whose pitch $\frac{m}{2n}=1$ [$m=2$, $n=1$], the general Möbius Surface having for its generators

$$\frac{x-a\cos\theta}{\cos\theta\,\cos\frac{m}{2n}\theta}=\frac{y-a\sin\theta}{\sin\theta\,\cos\frac{m}{2n}\theta}=\frac{z}{\sin\frac{m}{2n}\theta}$$

By eliminating θ, we get for the Cartesian equation of the surface of the fourth degree

$$x^2\,(y^2-z^2)+y^4-2axyz-a\,y^2=0$$

We shall call this equation, $\phi\,(x, y, z, a)=0$

The surface may be generated by a straight line which intersects the circle $z=0$, $x^2+y^2=a^2$, and moves so that the angle made by its projection on the plane of the circle with the initial radius is equal to the angle which it makes with the same projection (Fig. I). *IP* is a generator. In vectors, the equation of the General Möbius Surface is

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[*] Dr. C. E. Cullis, "On the Equations of the Möbius Surface of all pitches," *Bulletin of the Calcutta Mathematical Society*, Vol. I, pp. 163–186, 245–264.

[†] Dr. C. E. Cullis, "On a certain Cubic Surface called the Möbius Surface," *Bulletin of the Calcutta Mathematical Society*, Vol. I, pp. 9–30, 83–98.

$$\rho = (\alpha \cos t + \beta \sin t)\,(a + u) + \gamma .\, u \tan \frac{m}{2n} t$$

where α, β, γ are rectangular unit vectors; t, u are scalar parameters; a, $\frac{m}{2n}$ are constants.

For the surface of the fourth degree, $\frac{m}{2n} = 1$

2. SINGULARITIES. DOUBLE LINES ON THE SURFACE.

The double points are given by $\frac{\partial \phi}{\partial x} = 0$. $\frac{\partial \phi}{\partial y} = 0$, $\frac{\partial \phi}{\partial z} = 0$ which give respectively

$$2x\,(y^2 - z^2) - 2ayz = 0$$
$$2x^2 y + 4y^3 - 2axz - 2a^2 y = 0$$
$$-2x^2 z - 2axy = 0$$

These are satisfied by

$$(i)\ y = 0,\ z = 0$$
$$(ii)\ x = 0,\ y = 0$$

so that both the x-axis and the z-axis are double lines, the former only being a generator.

3. SECTIONS OF THE MÖBIUS SURFACE $\phi\,(x, y, z, a) = 0$ BY PLANES PARALLEL TO THE CO-ORDINATE PLANES.

Section by $x = k$.

$$y^4 + y^2\,(k^2 - a^2) - 2akyz - k^2 z^2 = 0.$$

The origin is a node, the tangents at which are

$$y^2\,(k^2 - a^2) - 2akyz - k^2 z^2 = 0$$

$$\text{i.e. } \frac{y}{z} = \frac{k}{k - a} \text{ and } -\frac{k}{k + a}$$

The curve meets the y-axis, i.e. $z = 0$ where $y = 0$ or $y^2 = a^2 - k^2$; the latter gives real values of y when $k < a$.

The origin is a point of inflexion, $\frac{d^2 y}{dx^2}$ which in terms of y is equal to $\frac{-k^2 y\,(2y^2 + 3k^2)}{[2y^2 + k^2 + a\sqrt{k^2 + y^2}]^3}$, vanishing when $y = 0$.

The origin is thus a biflecnode, a node at which both the tangents are stationary tangents The two asymptotes meeting the curve at three points at infinity are $y = \pm \infty$

The forms of the section when $k<a$, $k=a$, $k>a$ are shown in Figs. II, III and IV respectively.

Section by $y=k$.

$$x^2z^2+(2akxz-k^2x^2)+a^2k^2-k^4=0$$

The curve does not meet $x=0$ at any finite distance; it meets $z=0$ at the origin when $k=a$ and at two real points $\pm\sqrt{a^2-k^2}$ when $k<a$ The asymptotes meeting the curve at three points at infinity are $x=0$, $z=\pm k$

There is no finite double point, the node being at infinity.

When $k=a$, the equation of the section is $x\,[xz^2+a^2(2z-x)]=0$, and the origin is a point of inflexion.

The forms of the section when $k<a$, $k=a$ are shown in Figs. V and VI respectively.

Section by $z=k$.

$$y^2(x^2+y^2)-(kx+ay)^2=0$$

The curve meets $y=0$ at the origin, and $x=0$ at $y=0$ or $y=\pm a$.

The origin is a double point, a tacnode, the tangent at which is $kx+ay=0$, and meets the section in four points. The asymptotes meeting the curve at three points at infinity are $y=\pm k$.

The form of the section is shown in Fig. VII.

4. TACNODAL LINE AND BIFLECNODAL LINE.

It follows from the foregoing considerations that the surface cuts itself along a straight line, viz. the x-axis, the points on which form a series of biflecnodes; while two portions of the surface touch along another straight line, viz. the z axis, points on which are a series of tacnodes.

These properties enable us to give distinct names to the two double lines: the z-axis may be called the *tacnodal line*, and the x axis may be called the *biflecnodal line*

5. SECTION OF THE MÖBIUS SURFACE $\phi(x, y, z, a)=0$ BY ANY PLANE.

We shall investigate expressions for the co-ordinates of points on a section made by any plane.

Let the plane be $lx+my+nz-p=0$. Let us transform our co-ordinate axes, and let us take for our new origin the foot of the perpendicular drawn to the plane from the centre of the guiding circle, and for our new z-axis, the normal to the plane. The other two axes may be conveniently chosen by taking for our new x-axis the line on the plane perpendicular to the tac line.

Then if l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the new x-axis and the new y-axis respectively referred to the old axes, we have

$$0.\,l_1 + 0.\,m_1 + 1.\,n_1 = 0$$

$$ll_1 + mm_1 + nn_1 = 0$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$ll_2 + mm_2 + nn_2 = 0.$$

These give

$$l_1 = -\frac{m}{v},\quad m_1 = \frac{l}{v},\quad n_1 = 0,$$

$$l_2 = \frac{-nl}{v},\quad m_2 = -\frac{mn}{v},\quad n_2 = v,$$

when

$$v = \sqrt{l^2 + m^2} = \sqrt{1 - n^2}.$$

$\therefore$ the transformation scheme is

	x'	y'	$z'+p$
x	$-\frac{m}{v}$	$-\frac{nl}{v}$	l
y	$\frac{l}{v}$	$\frac{-mn}{v}$	m
z	0	v	n

$\therefore$ the equation of any generator when referred to the new axes is

$$\frac{-\frac{m}{v}x' - \frac{nl}{v}y' + l(z'+p) - a\cos\theta}{\cos\theta} = \frac{\frac{l}{v}x' - \frac{mn}{v}y' + m(z'+p) - a\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{vy' + n(z'+p)}{\sin\theta}.$$

This meets $z'=0$ in points determined (on simplification) by

$$vx' = \frac{(l\sin\theta - m\cos\theta)(an\sin\theta + p\cos\theta)}{(l\cos\theta + m\sin\theta)\cos\theta + n\sin\theta},$$

$$vy' = \frac{-(l\cos\theta + m\sin\theta)(a\sin\theta + np\cos\theta) + v^2p\sin\theta}{(l\cos\theta + m\sin\theta)\cos\theta + n\sin\theta}.$$

6. PLANE SECTIONS OF THE SURFACE $\phi(x, y, z, a)=0$.

The plane sections of the surface will have the same properties and the same characteristics as the corresponding sections of the simpler surface, the Möbius Cone of the Fourth Degree, the equation for which is obtained by putting $a=0$ in $\phi(x, y, z, a)=0$. In classifying the sections, we shall work with the simpler surface.

Möbius Cone of the Fourth Degree is the surface generated by straight lines

$\frac{x}{\cos^2\theta}=\frac{y}{\sin\theta\cos\theta}=\frac{z}{\sin\theta}$ where θ has all values from 0° to 360°. Its Cartesian Equation is $x^2(y^2-z^2)+y^4=0$. We shall call this equation $\psi(x, y, z)=0$.

7. CLASSIFICATION OF PLANE SECTIONS OF THE MÖBIUS CONE $\psi(x, y, z)=0$.

The plane sections of the Möbius Cone are most conveniently classified according to the nature of their asymptotes.

Now, the generators lying in any plane central section of the Möbius Cone are parallel to the asymptotes of its parallel plane sections. Our classification will therefore depend upon the nature of generators which lie on plane central sections.

8. GENERATORS LYING ON ANY CENTRAL SECTION OF THE CONE $\psi(x, y, z)=0$.

Let $lx+my+nz=0$ be any plane through the centre.

Let the generator θ lie on it, then $l\cos^2\theta+m\sin\theta\cos\theta+n\sin\theta=0$.

Let $\tan\frac{\theta}{2}=t$,

then we get
$$l(1-t^2)^2+2mt(1-t^2)+2nt(1+t^2)=0.$$
$$\text{i.e. } lt^4-2(m-n)t^3-2lt^2+2(m+n)t+l=0. \qquad (1)$$

This biquadratic gives the directions of the generators lying on any plane.

Since the expression on the right hand side becomes $4n$ when $t=1$ and $-4n$ when $t=-1$, the equation gives one, and therefore two, real roots. Therefore we shall always have two real generators lying on any plane central section. If t_1, t_2, t_3, t_4 are the roots of (1), they are connected by some pretty relations:—

$$\left.\begin{aligned} t_1 + t_2 + t_3 + t_4 &= \frac{2(m-n)}{l} \\ t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 + t_1t_3 + t_2t_4 &= -2 \\ t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 &= -\frac{2(m+n)}{l} \\ t\,t_2t_3t_4 &= 1. \end{aligned}\right\} \qquad (1a)$$

We may simplify our discussion of the biquadratic, by splitting it up into two convenient factors with the aid of some of these relations.

Since we have always at least two real generators on any central plane section, we may always choose two real quantities α and β, such that $t^2 - \beta t + \alpha$ is one of the factors of the biquadratic, β^2 being $\nless 4\alpha$.

Then $$\alpha = t_1t_2,\ \beta = t_1 + t_2.$$

With the aid of some of the relations (1*a*), we have, by putting $t_3t_4 = \gamma$, $t_3 + t_4 = \delta$,

$$\beta.\delta + \alpha + \gamma + 2 = 0,$$
$$\alpha.\gamma = 1,$$

or, $$\beta.\delta + \gamma + (\alpha + 2) = 0.$$
$$\alpha.\gamma - 1 = 0,$$

from which we get $$\delta = -\frac{(1+\alpha)^2}{\alpha\beta},$$
$$\gamma = \frac{1}{\alpha};$$

so that the other factor of the biquadratic is

$$t^2 + \frac{(1+\alpha)^2}{\alpha\beta}\,t + \frac{1}{\alpha}.$$

Therefore, *for all central sections, the biquadratic reduces to*

$$(t^2 - \beta t + \alpha)\left(t^2 + \frac{\overline{1+\alpha}^{\,2}}{\alpha\beta}\,t + \frac{1}{\alpha}\right) = 0, \qquad (2)$$

where α and β are functions of the directions of two real generators of the cone lying on the plane, $\alpha = t_1t_2$, $\beta = t_1 + t_2$

9. EQUATION OF THE PLANE IN TERMS OF α AND β, AND DEDUCTION OF A FORM OF GENERAL EQUATION FOR THE PLANE.

Before we proceed to discuss the biquadratic let us put the equation of any plane in terms of α and β, α and β having the meanings assigned above.

The direction cosines of a plane containing two generators of the cone t_1t_2 are determined by

$$l\,(1-t_1^2)+2mt_1\,(1-t_1^2)+2nt_1\,(1+t_1^2)=0$$
$$l\,(1-t_2^2)+2nt_2\,(1-t_2^2)+2nt_2\,(1+t_2^2)=0$$

These give, when we put $t_1t_2=\alpha$, $t_1+t_2=\beta$,

$$\frac{l}{4\alpha\beta}=\frac{m}{(\alpha^2-1)\,(\alpha+1)+\beta^2\,(\alpha-1)}=\frac{n}{(\alpha+1)^3-\beta\,(\alpha+1)}.$$

Therefore the equation of a central plane in terms of α and β (functions of the directions of two real generators lying on it) is

$$4\alpha\beta x+[(\alpha^2-1)\,(\alpha+1)+\beta^2\,(\alpha-1)]\,y+[(\alpha+1)^3-\beta^2\,(\alpha+1)]\,z=0.$$

The generators lying on this plane are given by (2).

Now as two real generators (determined by $t_1t_2=\alpha$, $t_1+t_2=\beta$) be on any central plane, we can *always* choose two *real* quantities α and β ($\beta^2 \nless 4\alpha$) so that,

$$\frac{l}{4\alpha\beta}=\frac{m}{(\alpha^2-1)\,(\alpha+1)+\beta^2\,(\alpha-1)}=\frac{n}{(\alpha+1)^3-\beta^2\,(\alpha+1)},$$

for *all* values of l, m and n.

Therefore the equation of *every* central plane can be put into the form,

$$4\alpha\beta x+[(\alpha^2-1)\,(\alpha+1)+\beta^2\,(\alpha-1)]\,y+[(\alpha+1)^3-\beta^2\,(\alpha+1)]\,z=0,$$

α and β being real quantities, $\beta^2 \nless 4\alpha$.

It follows that the *general equation of a plane* may be put into the form,

$$4\alpha\beta x+[(\alpha^2-1)\,(\alpha+1)+\beta^2\,(\alpha-1)]\,y+[(\alpha+1)^3-\beta^2\,(\alpha+1)]\,z+c=0, \qquad (3)$$

where α, β and c may receive all real values, provided $\beta^2 \nless 4\alpha$.

We may now regard our biquadratic (2) *as giving directions of generators lying on the general plane* (3) *when* $c=0$.

10. CLASSIFICATION OF ROOTS OF THE BIQUADRATIC (2).

The first quadratic gives $t=\frac{1}{2}\,[\beta\pm\sqrt{\beta^2-4\alpha}]$.

The second gives $t=\dfrac{1}{2\,\alpha\beta}\left[-(1+\alpha)^2\pm2\sqrt{\alpha\left\{\dfrac{(1+\alpha)^4}{4\alpha}-\beta^2\right\}}\,\right]$.

With regard to the latter, we shall have to deal separately with the cases when α is +ve and when α is −ve.

We have seen that there are always two real generators lying on any central plane; we may therefore proceed on the supposition that $\beta^2-4\alpha \nless 0$.

Now we may have the following cases :—

(*a*). *Four roots real and distinct.*

$$\beta^2 - 4\alpha > 0$$

$$\text{and when } \alpha \text{ is } +\text{ve, } \frac{(1+\alpha)^4}{4\alpha} > \beta^2,$$

$$\text{or when } \alpha \text{ is } -\text{ve, } \frac{(1+\alpha)^4}{4\alpha} < \beta^2.$$

These give (*i*) when α is +ve, $\frac{(1+\alpha)^4}{4\alpha} > \beta^2 > 4\alpha$,

(*ii*) when α is −ve, $\beta^2 >$ both $\frac{(1+\alpha)^4}{4\alpha}$ and 4α.

(*b*). *Two roots equal, the other two real and distinct.*

We have the following cases :—

I. $\beta^2 > 4\alpha$, $\frac{(1+\alpha)^4}{4\alpha} = \beta^2$,

II. $\beta^2 = 4\alpha$. when α is +ve, $\frac{(1+\alpha)^4}{4\alpha} > \beta^2$,

III. $\beta^2 = 4\alpha$, when α is −ve, $\frac{(1+\alpha)^4}{4\alpha} < \beta^2$.

(This is evidently an impossible case, since α = −ve does not satisfy the first equation here.)

These give (*i*) $\alpha = -1$ or $\not> 0$, $\frac{(1+\alpha)^4}{4\alpha} > 4\alpha$, $\beta^2 = \frac{(1+\alpha)^4}{4\alpha}$;

(*ii*) when α is + ve, $\frac{(1+\alpha)^4}{4\alpha} > 4\alpha$, $\beta^2 = 4\alpha$.

(*c*). *Two pairs of equal roots.*

$$\beta^2 = 4\alpha, \quad \frac{(1+\alpha)^4}{4\alpha} = \beta^2.$$

These give, when α is + ve, $\frac{(1+\alpha)^4}{4\alpha} = 4\alpha = \beta^2$

(*d*). *The two quadratics are the same, in other words, give the same roots. These also give two pairs of equal roots.*

From (2) Art. 8, $\alpha = \frac{1}{\alpha}$, $\frac{(1+\alpha)^4}{\alpha\beta} = -\beta$

$$\therefore \alpha = \pm 1, \quad \beta^2 = -\frac{(1+\alpha)^2}{\alpha}.$$

The real roots are $\alpha = -1$, $\beta = 0$.

(*e*). *Three roots equal.*

(*i*) Suppose $\beta^2 - 4\alpha = 0$.

Then we must have one of the following two relations true (either with the positive sign or with the negative sign) :—

$$\beta = \frac{-(1+\alpha)^2}{\alpha\beta} \pm \sqrt{\frac{(1+\alpha)^4}{\alpha^2\beta^2} - \frac{4}{\alpha}}$$

Simplifying and putting $\beta^2 = 4\alpha$, we see that the relation with the positive sign will be satisfied.

Thus we get

$$4\alpha^2 + (1+\alpha)^2 = +\sqrt{(1+\alpha)^4 - 16\alpha^2}.$$

This is satisfied if $\alpha = 0$ and therefore $\beta = 0$ also

(*ii*) Suppose $\beta^2 = \dfrac{(1+\alpha)^4}{4\alpha}$.

Proceeding as in the other case. we see that the following relation (with the negative sign) will be satisfied:—

$$-(1+\alpha)^2 - 4 = -\sqrt{(1+\alpha)^4 - 16\alpha^2}.$$

Dividing out by α^2 we get

$$-\left(1+\frac{1}{\alpha}\right)^2 - \frac{4}{\alpha^2} = -\sqrt{\left(1+\frac{1}{\alpha}\right)^4 - \frac{16}{\alpha^2}}$$

This is satisfied if $\alpha = \propto$ and therefore β also $= \propto$.

(*f*) *Two roots real and distinct, the other two imaginary.*

$$\beta^2 - 4\alpha > 0,$$

or, when α is + ve, $\quad \dfrac{(1+\alpha)^4}{4\alpha} < \beta^2,$

and when α is − ve, $\quad \dfrac{(1+\alpha)^4}{4\alpha} > \beta^2.$

This evidently is an impossible case since $\alpha = -$ ve does not satisfy

$$\frac{(1+\alpha)^4}{4\alpha} > \beta^2.$$

These therefore give, α + ve, $\beta^2 >$ both $\dfrac{(1+\alpha)^4}{4\alpha}$ and 4α.

(*g*). *Two roots equal, the other two imaginary.*

$$\beta^2 - 4\alpha = 0,$$

or, when α is + ve, $\quad \dfrac{(1+\alpha)^4}{4\alpha} < \beta^2,$

and when α is − ve, $\quad \dfrac{(1+\alpha)^4}{4\alpha} > \beta^2.$

This is evidently an impossible case since $\alpha = -$ ve does not satisfy $\beta^2 = 4\alpha$.

These therefore give, a + ve, $\frac{(1+a)^4}{4a} \angle 4a$.

N.B. We cannot have four roots equal, as the real roots of (*d*) do not satisfy (*c*).

Before we proceed to simplify these results, and to see which are possible cases and which not, we may draw a graph of the function $\frac{(1+a)^4}{4a} - 4a$, which is of frequent recurrence in the aforesaid results.

The form of the graph is shown in Fig. VIII.

We may consider the properties of the graph.

(*i*) $$\frac{(1+a)^4}{4a} - 4a = 0,$$

when $(1-a)^2\,[1+6a+a^2] = 0$

or when $a = 1$ or $a = \frac{-6 \pm \sqrt{36-4}}{2} = -3 \pm 2\sqrt{2} = -\cdot172$ or $-5\cdot828$

(*ii*) Putting $\frac{(1+a)^4}{4a} - 4a = \Delta$

$$\frac{d\Delta}{da} \text{ becomes} = 0 \text{ when } 3a^4 + 8a^3 - 10a^2 - 1 = 0$$

One of the roots of this biquadratic is $a = 1$ and taking out $a - 1$ as a factor, the resulting cubic is $3a^3 + 11a^2 + a + 1 = 0$, the real roots of which, if any, must clearly be negative.

Let $3a^3 + 11a^2 + a + 1 = f(a)$.

Sturm's remainders for this function are

$$f_1(a) = 9a^2 + 22a + 11,$$
$$f_2(a) = 121a - 8,$$
$$f_3(a) = -\text{ve}.$$

So that there is only one negative real root.

(*iii*) When $a = 0$, $\Delta = \infty$

$$\text{When } a = -\cdot1,\ \Delta = \frac{(\cdot9)^4}{-\cdot4} + \cdot4 = -\frac{\cdot656}{\cdot4} + \cdot4 = -1\cdot24$$

$$\text{When } a = \cdot1,\ \Delta = \frac{(1\cdot1)^4}{\cdot4} - \cdot4 = +\text{ve}$$

We may now, with the help of the graph, resume our consideration of the roots of the biquadratic.

We find that (*g*) is an impossible case, for it is clear from the graph, that for positive values of a, $\frac{(1+a)^4}{4a} - 4a$ can never be negative.

We also get the real roots of (*c*) by a reference to the graph. As solutions of $\frac{(1+\alpha)^4}{4\alpha}=4\alpha$, the graph gives $\alpha=1$ or $-\cdot172$ or $-5\cdot828$. Therefore the roots of the equation, so that α is +ve, are $\alpha=+1$, $\beta=\pm2$.

11. Special cases in which Generators for different values of *t* (or *θ*) coincide.

Before we finally classify all the possible cases, we shall also consider the special cases in which the generators for two different non-consecutive values of *t* (or *θ*) coincide. These clearly occur in the cases of (*i*) planes through the *x*-axis; (*ii*) planes through the *z*-axis; and (*iii*) the *y*-plane which passes through both the axis of *x* and *z*.

(*i*) For planes through the *x*-axis,

$l=0$ but n must not $=0$,

i.e. $4\alpha\beta=0$, $(\alpha+1)^3-\beta^2(\alpha+1)\neq0$.

i.e. $\alpha=0$, $\beta\neq\pm1$,

or $\beta=0$ $\alpha\neq-1$

Our biquadratic may be put thus :—

$$(t^2-\beta t+\alpha)(\alpha\beta t^2+\overline{1+\alpha^2}\,t+\beta)=0,$$

When $\alpha=0$, this becomes $(t^2-\beta t)(0.\ t^2+t+\beta)=0$.

$$\therefore t=0,\ \beta,\ \alpha,\ -\beta,\ \text{provided } \beta\neq\pm1.$$

When $\beta=0$, this becomes $(t^2+\alpha)(0.\ t^2+\overline{1+\alpha^2}\,t+0)=0$.

$$\therefore t=+\sqrt{-\alpha},\ -\sqrt{-\alpha},\ 0,\ \propto,\ \text{provided } \alpha\neq-1.$$

This gives rise to two cases according as α is −ve or +ve.

We may compare case (*e*) (*i*), considered before, with this.

(*ii*) For planes through the *z*-axis,

$n=0$ but $l\neq0$,

i.e. $(\alpha+1)^3-\beta^2(\alpha+1)=0$, $\alpha\beta\neq0$,

i.e. $\alpha=-1$, $\beta\neq0$,

or $\beta=\pm(\alpha+1)$, $\alpha\neq0$, $\beta\neq0$.

When $\alpha=-1$, the biquadratic becomes $(t^2-\beta t-1)(t^2-1)=0$.

$$\text{i.e. } t=\pm1,\ \frac{\beta\pm\sqrt{\beta^2+4}}{2}$$

When $\beta=\pm(\alpha+1)$, the biquadratic becomes $(t-1)(t+1)(t-\alpha)(\alpha t+1)=0$,

$$\text{i.e. } t=\pm1,\ \alpha,\ -\frac{1}{\alpha}\ \text{provided } \alpha\neq0.$$

(*iii*) For the y-plane,

$l=0$, $n=0$ but $m \neq 0$,

i.e. $\alpha\beta=0$, $(\alpha+1)^3-\beta^2(\alpha+1)=0$, $(\alpha^2-1)(\alpha+1)+\beta^2(\alpha-1)\neq 0$.

The only admissible values are $\alpha=0$, $\beta \neq \pm 1$, since $\beta=0$, $\alpha=-1$ does not satisfy the third equation (of inequality).

$\therefore$ the biquadratic for t gives $t=0$, ∞, ± 1,

i.e. $\theta=0^\circ, 90^\circ, 180^\circ, 270^\circ$.

It may be noted that $\alpha=-1$, $\beta=0$ make not only $n=0$, $l=0$ but also $m=0$, and it may be interesting to determine this apparently indeterminate form. We may put $\beta=1+\alpha$, $\alpha=-1$ as equivalent to $\alpha= 1$, $\beta=0$. By substituting $\beta=1+\alpha$ in the biquadratic, we have

$$(t^2-\overline{1+\alpha}.t+\alpha)(\alpha.\overline{1+\alpha}.t^2+\overline{1+\alpha^2}.t+\overline{1+\alpha})=0$$

Taking out $(1+\alpha)$ as common factor, $(t^2-\overline{1+\alpha}.t+\alpha)(\alpha t^2+\overline{1+\alpha}.t+1)=0$

Putting $\alpha=-1$, we get $(t^2-1)(t^2-1)=0$.

This is therefore a singular case in which all the four roots are real, but they all give the same line for the generator.

It may be noted that this is case (*d*), and gives the same result as case (*c*).

12. DIFFERENT CLASSES OBTAINED FROM THE FOREGOING CONSIDERATIONS.

From the foregoing considerations, we conclude that we may have the following classes of plane central sections:—

1. Four generators real and distinct,

(*i*) α + ve $\dfrac{(1+\alpha)^4}{4\alpha} > \beta^2 > 4\alpha$,

(*ii*) α − ve, $\beta^2 >$ both $\dfrac{(1+\alpha)^4}{4\alpha}$ and 4α

2. Four real generators, of which two are consecutive and coincident

(*i*) $\beta^2 > 4\alpha$, $\dfrac{(1+\alpha)^4}{4\alpha}=\beta^2$.

(*ii*) α + ve. $\beta^2=4\alpha$, $\dfrac{(1+\alpha)^4}{4\alpha} > \beta^2$.

3. Two pairs of real consecutive coincident generators. It so happens, as we have already seen, that the generator for one pair coincides with that for the other.

$\alpha=1$. $\beta=\pm 2$ which gives the same result as $\alpha=-1$. $\beta=0$.

4. Three consecutive coincident generators coinciding with another, not consecutive.

(*i*) $\alpha=0, \beta=0$,

(*ii*) $\alpha=\propto, \beta=\propto$.

5. Two generators real and distinct, the other two imaginary,

$$\alpha + \text{ve}, \beta^2 \not< \text{both } 4\alpha \text{ and } \frac{(1+\alpha)^4}{4\alpha}.$$

6. A pair of real non-consecutive coincident generators ($\theta=0°$, $\theta=180°$), the other two real,

(*i*) $\alpha=0, \beta\neq\pm1$,

(*ii*) $\beta=0, \alpha - \text{ve but} \neq -1$.

7. A pair of real non-consecutive coincident generators ($\theta=0°$, $\theta=180°$), the other two imaginary,

$$\beta=0, \quad \alpha + \text{ve}.$$

8. A pair of real non-consecutive coincident generators ($\theta=90°$, $\theta=270°$), the other two real,

(*i*) $\alpha=-1, \beta\neq0$,

(*ii*) $\beta=\pm(\alpha+1)$, $\alpha\neq0, \beta\neq0$.

9. Two pairs of real non-consecutive coincident generators,

$$\alpha=0, \beta=\pm1.$$

The central sections of the cone having been thus distributed into nine classes according to the character of the generators lying on them, all sections of the cone can be distributed into as many classes according to the character of their asymptotes. One section of the cone for each of these classes has been drawn. In tracing the sections, use has been made of the values of co-ordinates obtained in Art. 5.

Fig. IX is a section of class 1 ($\alpha=-3, \beta=1$).
Fig. X is a section of class 2 ($\alpha=4, \beta=4$).
Fig. XI is a section of class 3 ($\alpha=-1, \beta=0$) [asymptotes : $x=\pm\propto$].
Fig. XII is a section of class 4 ($\alpha=\propto, \beta=\propto$).
Fig. XIII is a section of class 5 ($\alpha=1, \beta=3$).
Fig. XIV is a section of class 6 ($\alpha=0, \beta=2$).
Fig. XV is a section of class 7 ($\alpha=0, \beta=2$).
Fig. XVI is a section of class 8 ($\alpha=-1, \beta=1$).
Fig. XVII is a section of class 9 ($\alpha=0, \beta=\pm1$).

13. LINES OF STRICTION OF THE SURFACE $\phi(x, y, z, a)=0$

The direction cosines of two consecutive generators θ and $\theta+d\theta$ are $\cos^2\theta$, $\sin\theta\cos\theta$, $\sin\theta$,

and $$\cos^2(\theta+d\theta),\ \sin(\theta+d\theta)\cos(\theta+d\theta),\ \sin(\theta+d\theta)$$

or $\cos^2\theta-2\sin\theta\cos\theta\, d\theta$, $\sin\theta\cos\theta+(\cos^2\theta-\sin^2\theta)\, d\theta$, $\sin\theta+\cos\theta\, d\theta$

$\therefore$ if λ, μ, ν be the direction cosines of the common perpendicular to θ and $\theta+d\theta$,

$$\lambda\cos^2\theta+\mu\sin\theta\cos\theta+\nu\sin\theta=0,$$

$$\lambda(\cos^2\theta-2\sin\theta\cos\theta d\theta)+\mu(\sin\theta\cos\theta+\overline{\cos^2\theta-\sin^2\theta}\, d\theta)$$
$$+\nu(\sin\theta+\cos\theta\, d\theta)=0.$$

These give $$\frac{\lambda}{\sin^3\theta}=\frac{\mu}{\cos\theta(\cos^2\theta-2)}=\frac{\nu}{\cos^2\theta}$$

Let p, q, r be direction cosines of the perpendicular to the generator θ and the common perpendicular, i.e. to the plane containing the generator θ and the common perpendicular to it and the next generator $\theta+d\theta$,

Then $$p\cos^2\theta+q\sin\theta\cos\theta+r\sin\theta=0,$$
$$p\sin^3\theta+q\cos\theta(\cos^2\theta-2)+r\cos^2\theta=0.$$

This gives $$\frac{p}{2\sin\theta\cos\theta}=\frac{q}{\sin^2\theta-\cos^2\theta}=\frac{r}{-\cos\theta}.$$

Similarly if p', q', r' be the direction cosines of the perpendicular to the generator $\theta+d\theta$ and the common perpendicular, i.e. to the plane containing the generator $\theta+d\theta$ and the common perpendicular to it and the generator θ,

$$\frac{p'}{2\sin\theta\cos\theta+\cos^2\theta(\cos^2\theta+1)d\theta}=\frac{q'}{\sin^2\theta-\cos^2\theta+\sin\theta\cos\theta(1+\cos^2\theta)d\theta}$$
$$=\frac{r'}{-\cos\theta+\sin\theta(\cos^2\theta+1)d\theta}$$

Thus the common perpendicular is given as the intersection of the planes,

$$(\xi-a\cos\theta)2\sin\theta\cos\theta+(\eta-a\sin\theta)(\sin^2\theta-\cos^2\theta)-\zeta\cos\theta=0,$$

and $$(\xi-a\cos\theta+a\sin\theta\, d\theta)(2\sin\theta\cos\theta+\cos^2\theta.\overline{1+\cos^2\theta}\, d\theta)$$
$$+(\eta-a\sin\theta-a\cos\theta\, d\theta)(\sin^2\theta-\cos^2\theta+\sin\theta\cos\theta.\overline{1+\cos^2\theta}\, d\theta)$$
$$-\zeta(\cos\theta-\sin\theta.\overline{1+\cos^2\theta}\, d\theta)=0.$$

$\therefore$ the line of striction which is the locus of points when the common perpendicular meets the generator θ is given by these equations, and

$$\frac{\xi - a\cos\theta}{\cos^2\theta} = \frac{\eta - a\sin\theta}{\sin\theta\cos\theta} = \frac{\zeta}{\sin\theta}.$$

i.e. $(\xi - a\cos\theta)\cos^2\theta\,(\cos^2\theta + 1) + 2a\sin^2\theta\cos\theta + (\eta - a\sin\theta)\sin\theta\cos\theta\,(1 + \cos^2\theta)$
$- a\cos\theta\,(\sin^2\theta - \cos^2\theta) + \zeta\sin\theta\,(\cos^2\theta + 1) \quad 0.$

and
$$\frac{\xi - a\cos\theta}{\cos^2\theta} = \frac{\eta - a\sin\theta}{\sin\theta\cos\theta} = \frac{\zeta}{\sin\theta}.$$

These give
$$\xi = \frac{a\cos\theta}{1 + \cos^2\theta},$$
$$\eta = \frac{a\sin\theta}{1 + \cos^2\theta},$$
$$\zeta = \frac{-a\sin\theta\cos\theta}{1 + \cos^2\theta}.$$

From these we get
$$\frac{\zeta^2}{\eta^2} + \frac{\zeta^2}{\xi^2} = 1,$$
i.e. $(\xi^2 + \eta^2)\,\zeta^2 = \xi^2\eta^2$,
i.e. $(x^2 + y^2)\,z^2 = x^2y^2$, a cone of the fourth degree.

Thus the lines of striction are the intersections of this cone and the surface. This cone has a fine symmetrical form, as can be seen by considering its plane sections parallel to the co-ordinate planes.

Sections by planes parallel to the z=plane ($z=k$).

$$x^2y^2 - k^2(x^2 + y^2) = 0.$$

Its asymptotes are $x = \pm k$, $y = \pm k$.

It is also clear that x and y are interchangeable and that they cannot be numerically less than k. The origin is a conjugate point.

The form of the section is shown in Fig. XVIII.

Sections by a plane parallel to the x=plane ($x=k$).

$$(k^2 + y^2)\,z^2 = k^2y^2, \text{ or } y^2z^2 + k^2(z^2 - y^2) = 0.$$

Origin is a node, the tangents at which are $z+y=0$, $z-y=0$.

Its asymptotes are $z = \pm k$

The form of the section is shown in Fig. XIX.

Sections by planes $y=k$.

These are the same as sections by planes $x=k$.

14. ENVELOPES OF THE PROJECTIONS ON ANY PLANE OF THE GENERATORS OF THE SURFACE $\phi(x, y, z, a)=0$.

Let $lx+my+nz-p=0$ be any plane, and let it be required to find the envelope of the projections on it of the generators. Transforming the co-ordinate axes as in Art. 5, the new equation of the generators θ becomes

$$\frac{-\frac{m}{v}x' - \frac{nl}{v}\,y' + l\,(z' + p) - a\cos\theta}{\cos^2\theta} = \frac{\frac{l}{v}x' - \frac{mn}{v}\,y' + m\,(z' + p) - a\sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{vy' + n\,(z' + p)}{\sin\theta}.$$

$\therefore$ the equation of the projection of this generator on the given plane ($z'=0$), obtained by eliminating z' and after simplification, is

$$x\left[\frac{n}{v}\sin\theta\cos\theta\,(l\cos\theta + m\sin\theta) - (1 - n^2)\,\frac{\sin^2\theta}{v}\right]$$
$$+ y\left[\left(\frac{n^2}{v} + v\right)\sin\theta\cos\theta\,(l\sin\theta - m\cos\theta)\right] + a^2\sin^2\theta\,(l\sin\theta - m\cos\theta) = 0.$$

If we put $\tan\frac{\theta}{2} = t$, this becomes, when expressed in descending powers of t,

$$(lnx - my)\,t^4 + 2(-v^2x - mnx - ly + amv)\,t^3 + 2(-lnx + my + 2\,alv)\,t^2$$
$$+ 2(-v^2x + mnx + ly - amv)\,t + (lnx - my) = 0.$$

To find the envelope of such lines, we put $lnx - my = A$,

$$2\,(-v^2x - mnx - ly + amv) = B,$$
$$2\,(-lnx + my + 2\,alv) = C,$$
$$2\,(-v^2x + mnx + ly - amv) = D,$$

and eliminate t between the equations

$$At^4 + Bt^3 + Ct^2 + Dt + A = 0$$

and

$$4At^3 + 3Bt^2 + 2Ct + D = 0.$$

Now, adopting Cauchy's form of Bezout's method of elimination, we get the four equations

$$4At^3 + 3Bt^2 + 2Ct + D = 0,$$
$$Bt^3 + 2Ct^2 + 3Dt + 4A = 0.$$
$$2ACt^3 + (3AD + BC)\,t^2 + (4A^2 + 2BD)\,t + 3AB = 0,$$
$$3ADt^3 + (2BD + 4A^2)\,t^2 + (CD + 3AB)\,t + 2\,AC = 0.$$

Whence as resultant, we get the determinant

$$\begin{vmatrix} 4A & 3B & 2C & D \\ B & 2C & 3D & 4A \\ 2AC & 3AD + BC & 2BD + 4A^2 & 3AB \\ 3AD & 2BD + 4A^2 & CD + 3AB & 2AC \end{vmatrix} = 0.$$

If this determinant is expressed in descending powers A, we get the resultant in the form

$$256A^6 + 128A^4\,(BD - C^2) + 48A^3C\,(D^2 - B^2) + A^2\,(-32BC^2D - 114B^2D^2$$
$$+ 27B^4 + 27D^4 + 16C^4) + A\,(6BCD.\overline{B^2 - 3D^2} + 4C^3.\overline{D^2 - B^2})$$
$$+ B^2D^2\,(4BD - C^2) = 0.$$

15. ENVELOPE OF PROJECTIONS ON THE CO-ORDINATE PLANES OF THE GENERATOR OF THE SURFACE $\phi(x, y, z, a)=0$.

I. For plane $x=0$,

$$A=0,\ B=-2(x+y),\ C=4a,\ D=2(y-x).$$

$\therefore$ envelope is
$$16\,(y^2-x^2)^2\,[16\,(x^2-y^2)-16a^2]=0,$$
$$\text{i.e. } (x^2-y^2)^2\,(x^2-y^2-a^2)=0.$$

II. For plane $z=0$,

The envelope is the origin, as is also clear, the projection of

$$\frac{x-a\cos\theta}{\cos^2\theta}=\frac{y-a\sin\theta}{\sin\theta\cos\theta}=\frac{z}{\sin\theta}$$ on the $z=$ plane being $y=x\tan\theta$.

III. For plane $y=0$,

$$A=-y,\ B=-2(x-a),\ C=2y,\ D=-2(x+a)\,.$$

$\therefore$ envelope is

$$256y^6+128y^4\,[4\,(x^2-a^2)-4y^2]-96{\cdot}16y^4ax$$
$$+y^2\,[-64{\cdot}8y^2\,(x^2-a^2)-114{\cdot}16\,(x^2-a^2)^2+27{\cdot}16\,\{(x+a)^4+(x-a)^4\}+16{\cdot}16y^4]$$
$$-y\,[48y\,(x^2-a^2).4(\overline{x-a}^2-3.\overline{x+a})^2+32.16y^3ax]$$
$$+16(x^2-a^2)^2\,[16(x^2-a^2)-4y^2]=0.$$

Or, on simplification, in descending powers of y,

$$16axy^4+(5x^4-12ax^3-70a^2x^2+12a^3x+11a^4)y^2-2(x^2-a^2)^3=0.$$

When the origin is transferred to the point $(a, 0)$, the equation becomes

$$x^4\,(2x^2-5y^2)+4ax\,(3x^4-2x^2y^2-4y^4)+4a^2\,(6x^4+19x^2y^2-4y^4)$$
$$+16a^3x\,(x^2+9y^2)+54a^4y^2=0.$$

When the origin is transferred to the point $(-a, 0)$, the equation becomes

$$x^4\,(2x^2-5y^2)+4ax\,(8x^2y^2-4y^4-3x^4)+4a^2\,(6x^4+4y^4+x^2y^2)$$
$$-16a^3x\,(x^2+6y^2)+54a^4y^2=0.$$

(1) The curve is symmetrical with regard to the x-axis.

(2) It meets the x-axis at $x=\pm a$.

(3) The point $(a, 0)$ is a cusp, the tangent at this point being $y=0$, the character of the curve at this point being $x^3=-\frac{27}{8}ay^2$.

(4) The point $(-a, 0)$ is also a cusp, the tangent at this point being $y=0$, the character of the curve at this point being $x^3=\frac{27}{8}ay^2$.

(5) The asymptotes are

$$x=0,$$

$$y=\sqrt{\frac{2}{5}}\,x+\frac{14}{25}\sqrt{\frac{2}{5}}\,a,$$

$$y=\quad\sqrt{\frac{2}{5}}\,x-\frac{14}{25}\sqrt{\frac{2}{5}}\,a.$$

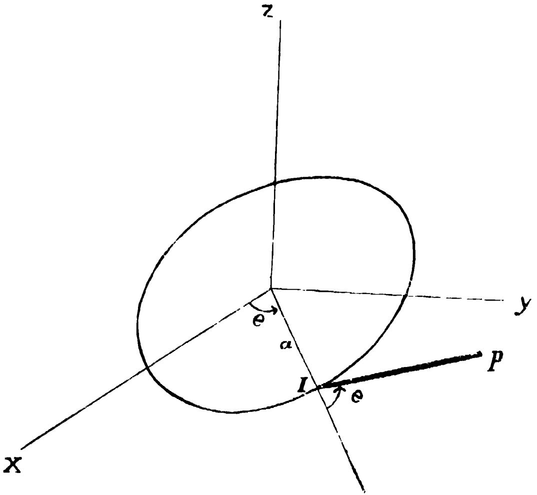


Fig. I

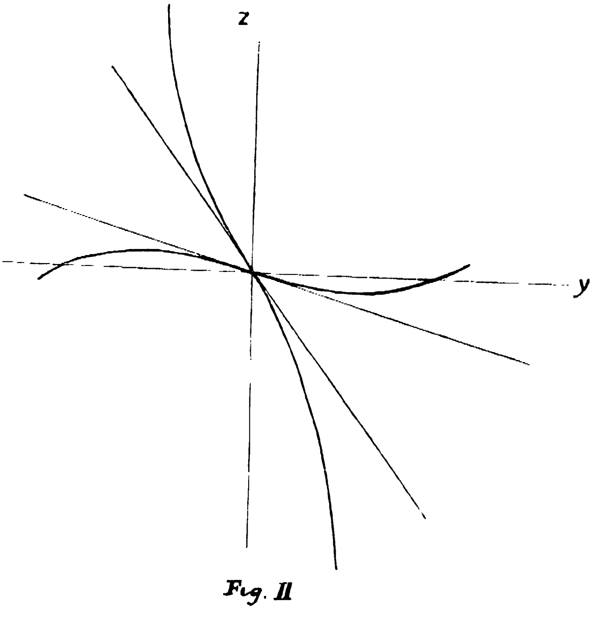


Fig. II

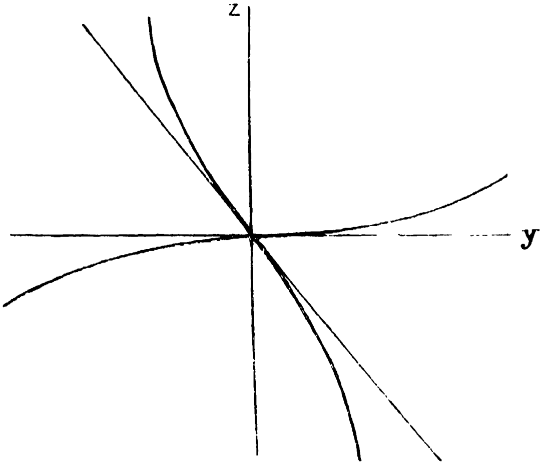
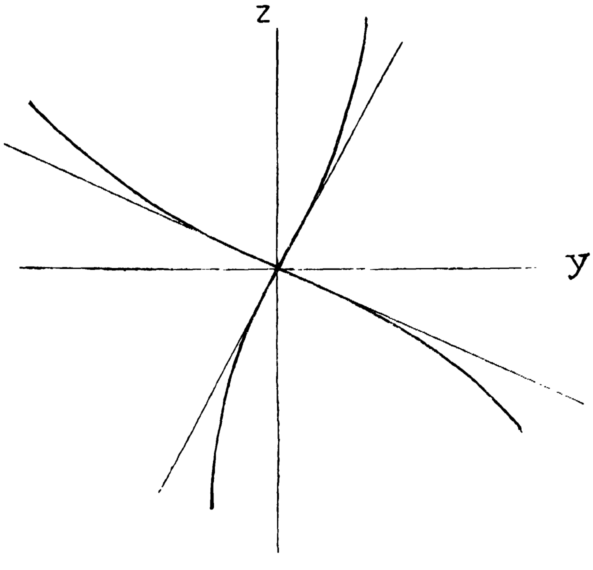


Fig. III



Fig, IV.

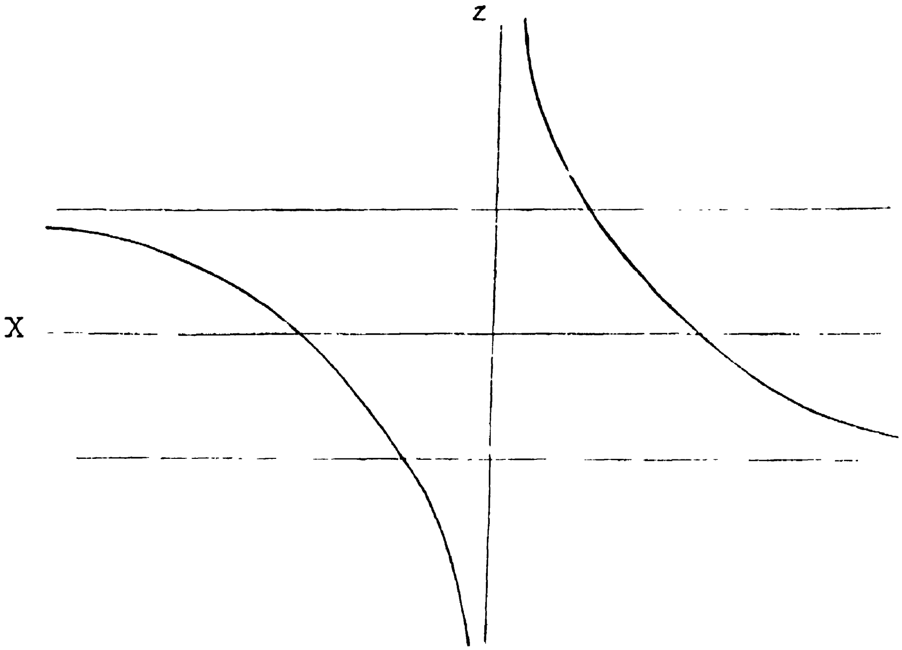


Fig. V.

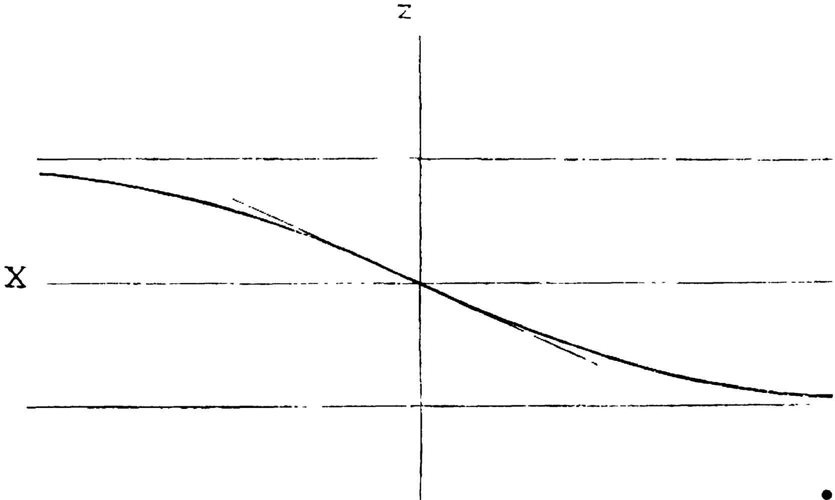


Fig VI.

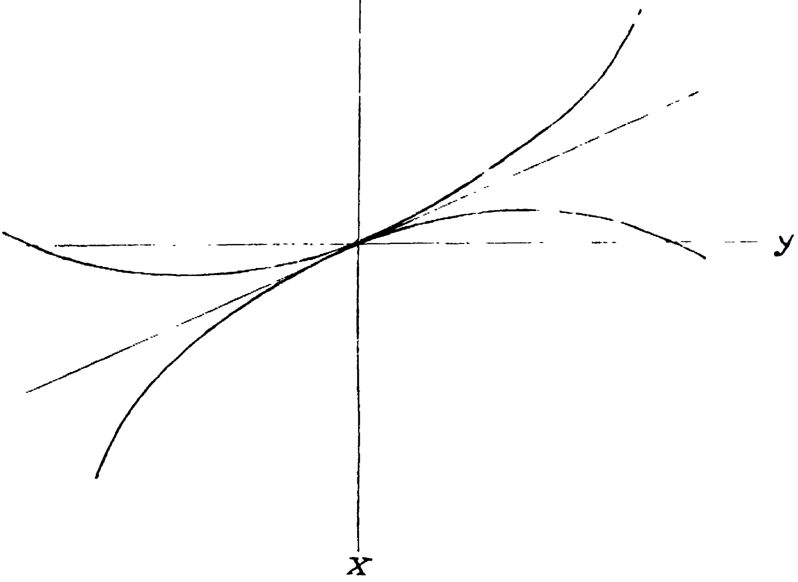


Fig. VII

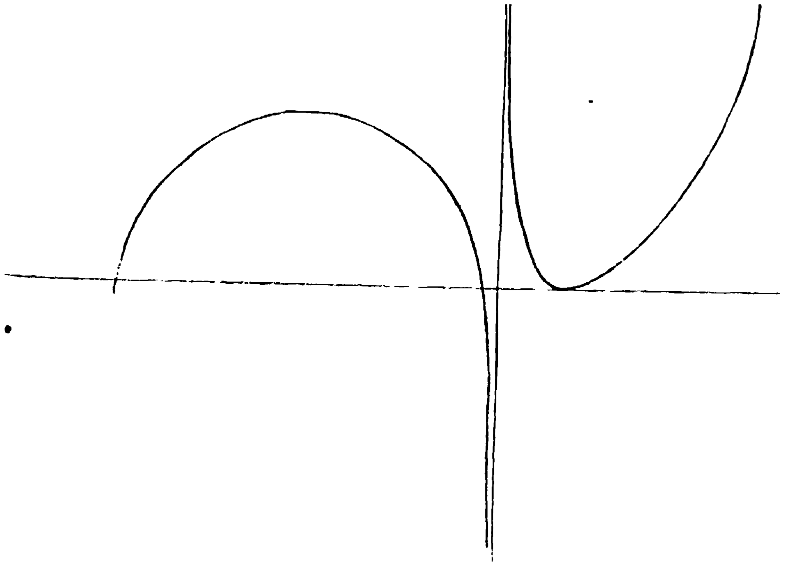


Fig. VIII

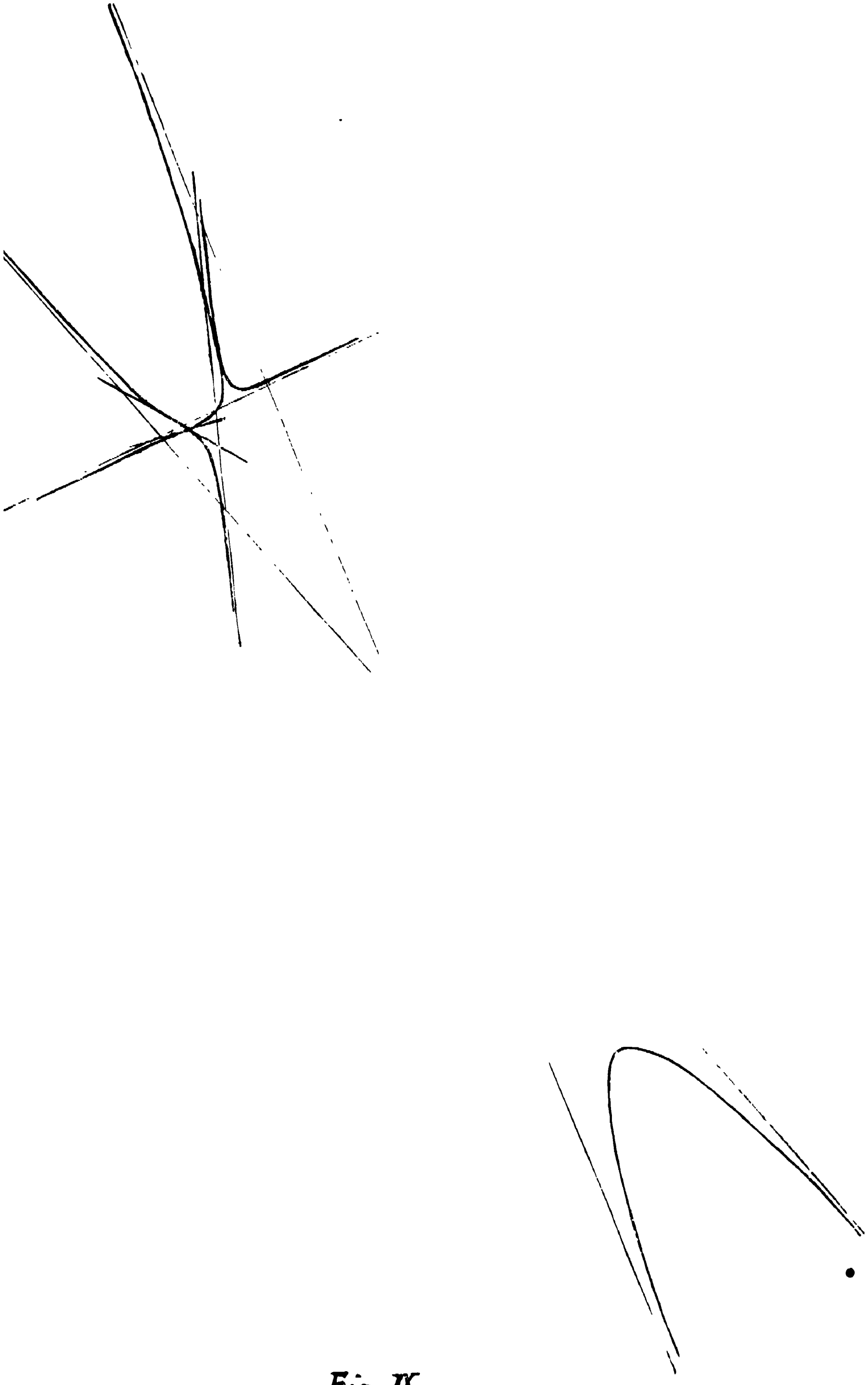


Fig. IX.

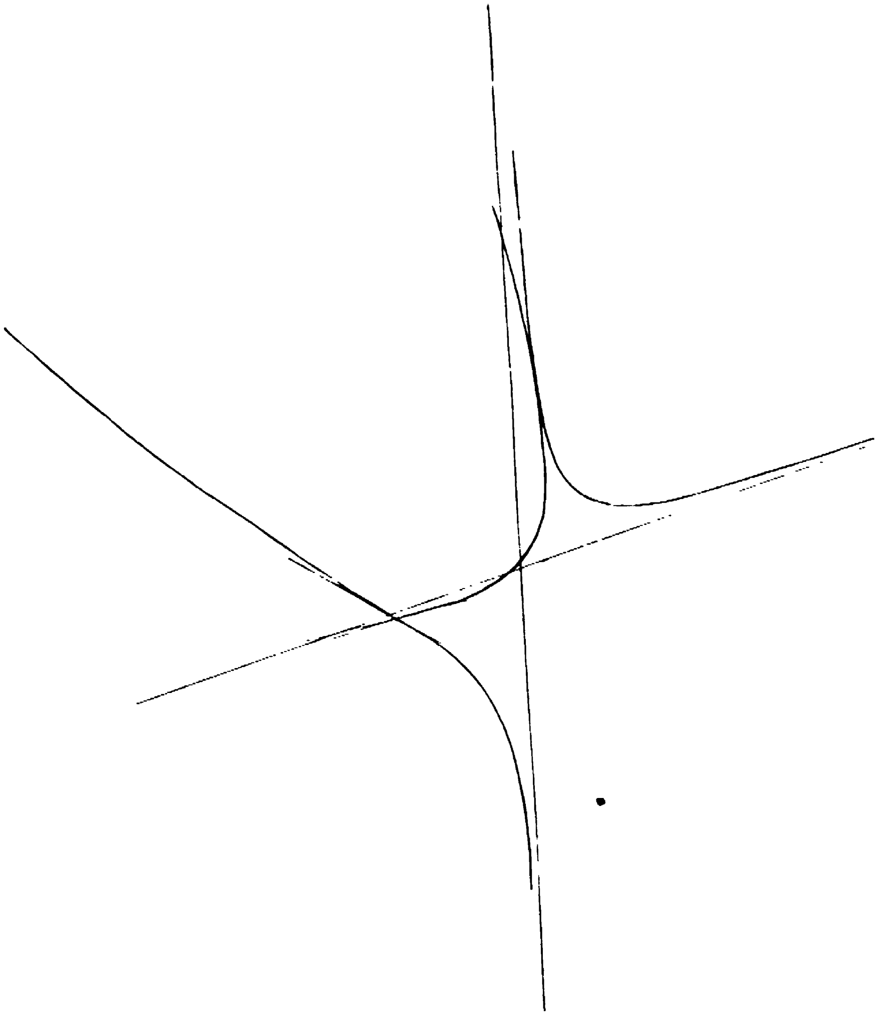


Fig. X.

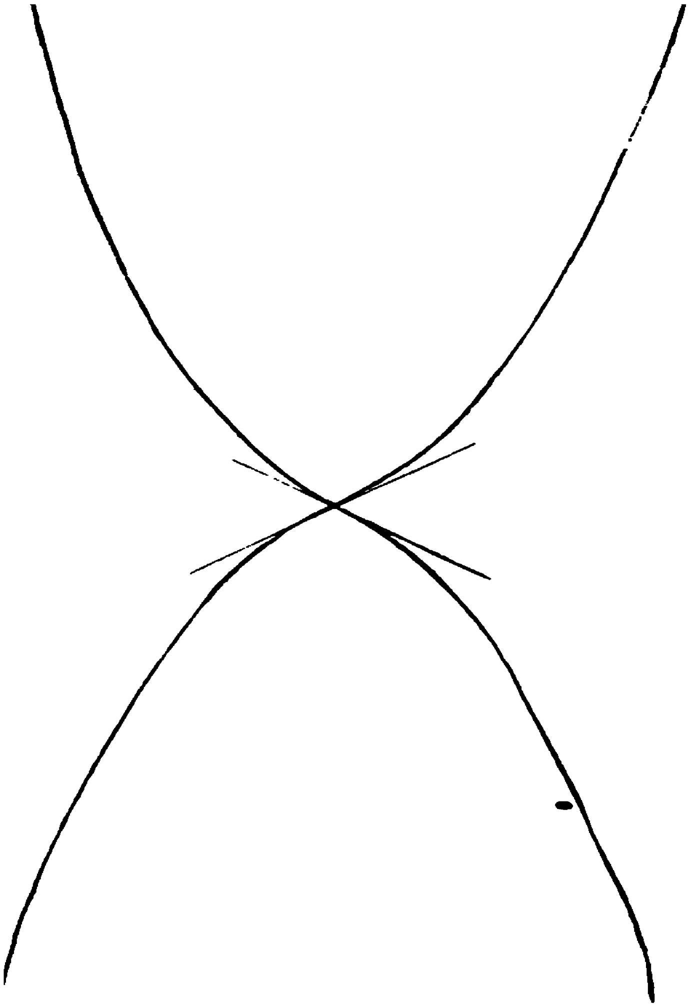


Fig. XI

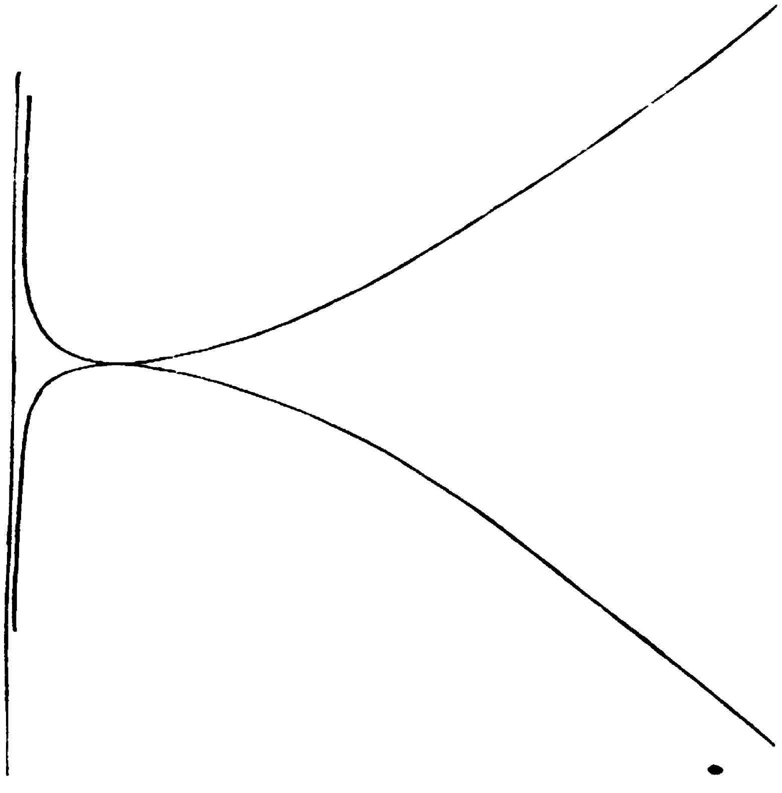


Fig XII

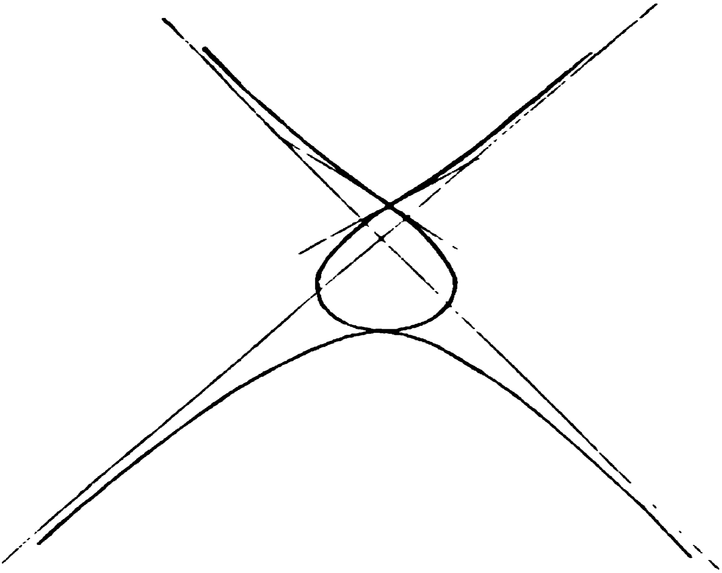


Fig. XIII

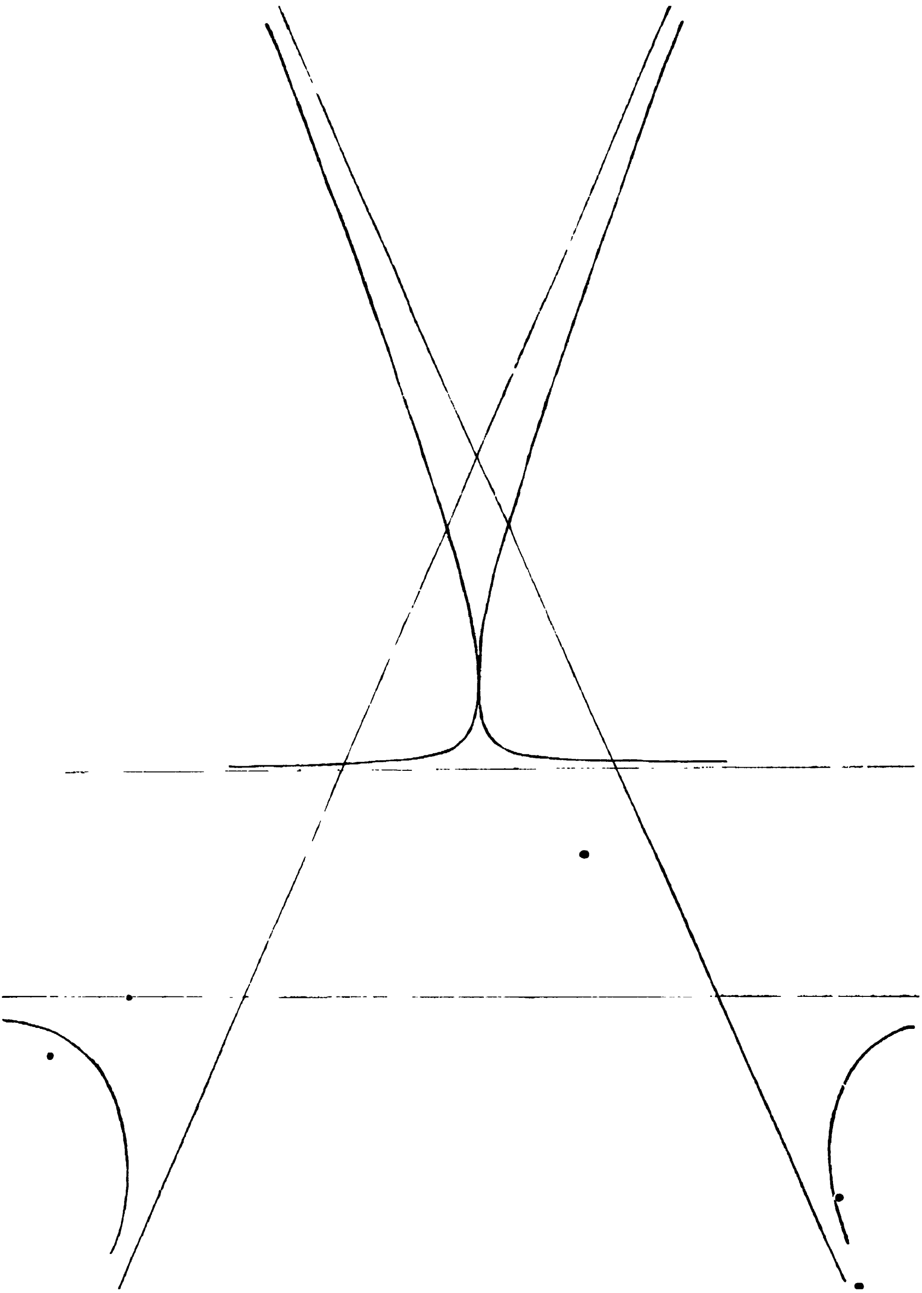


Fig. XIV

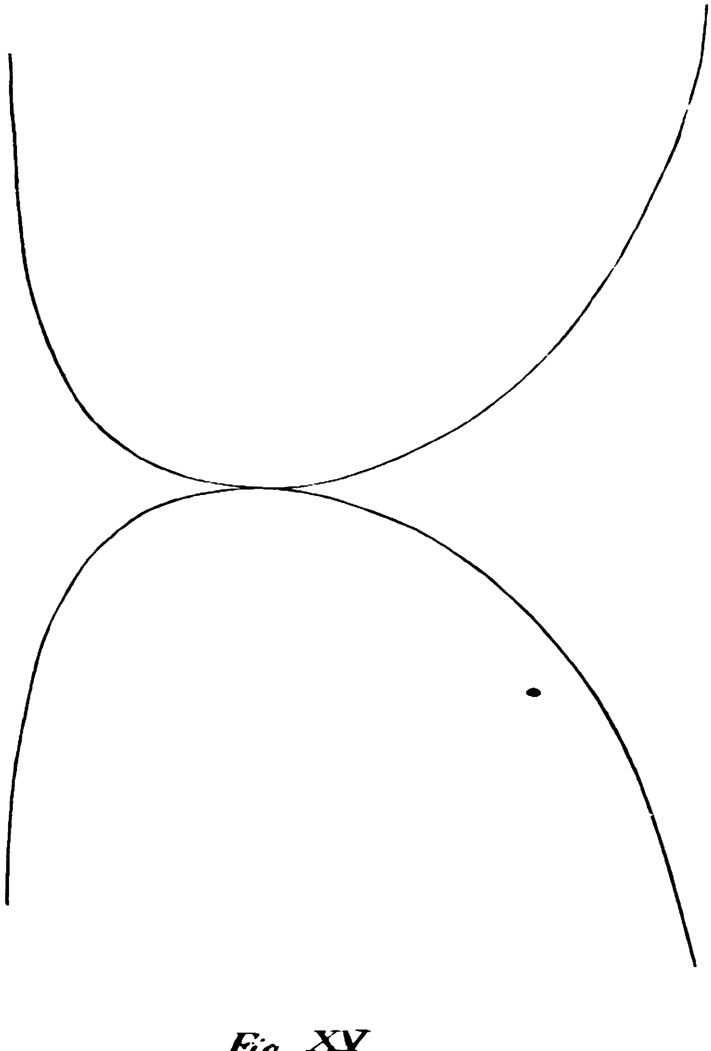


Fig XV

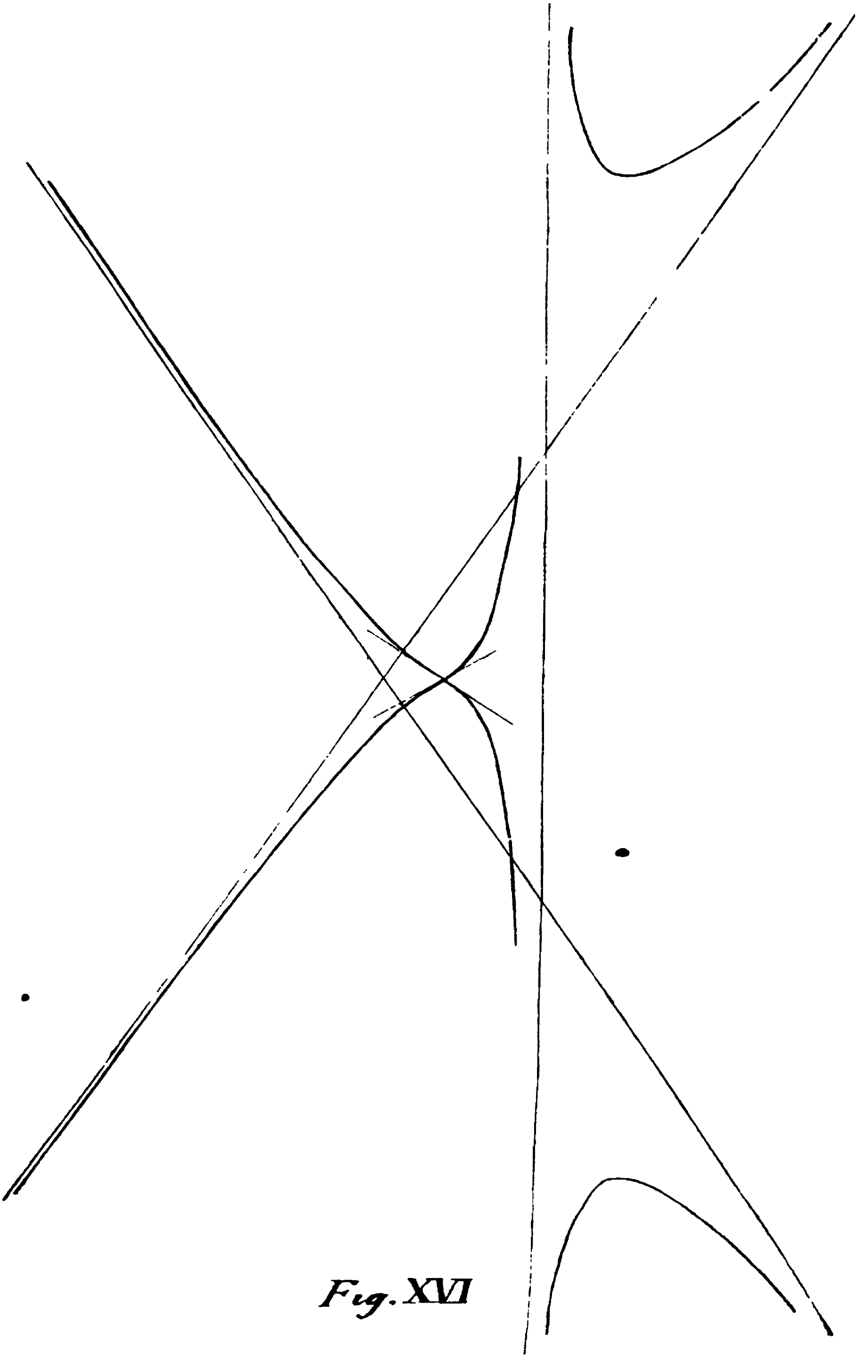


Fig. XVI

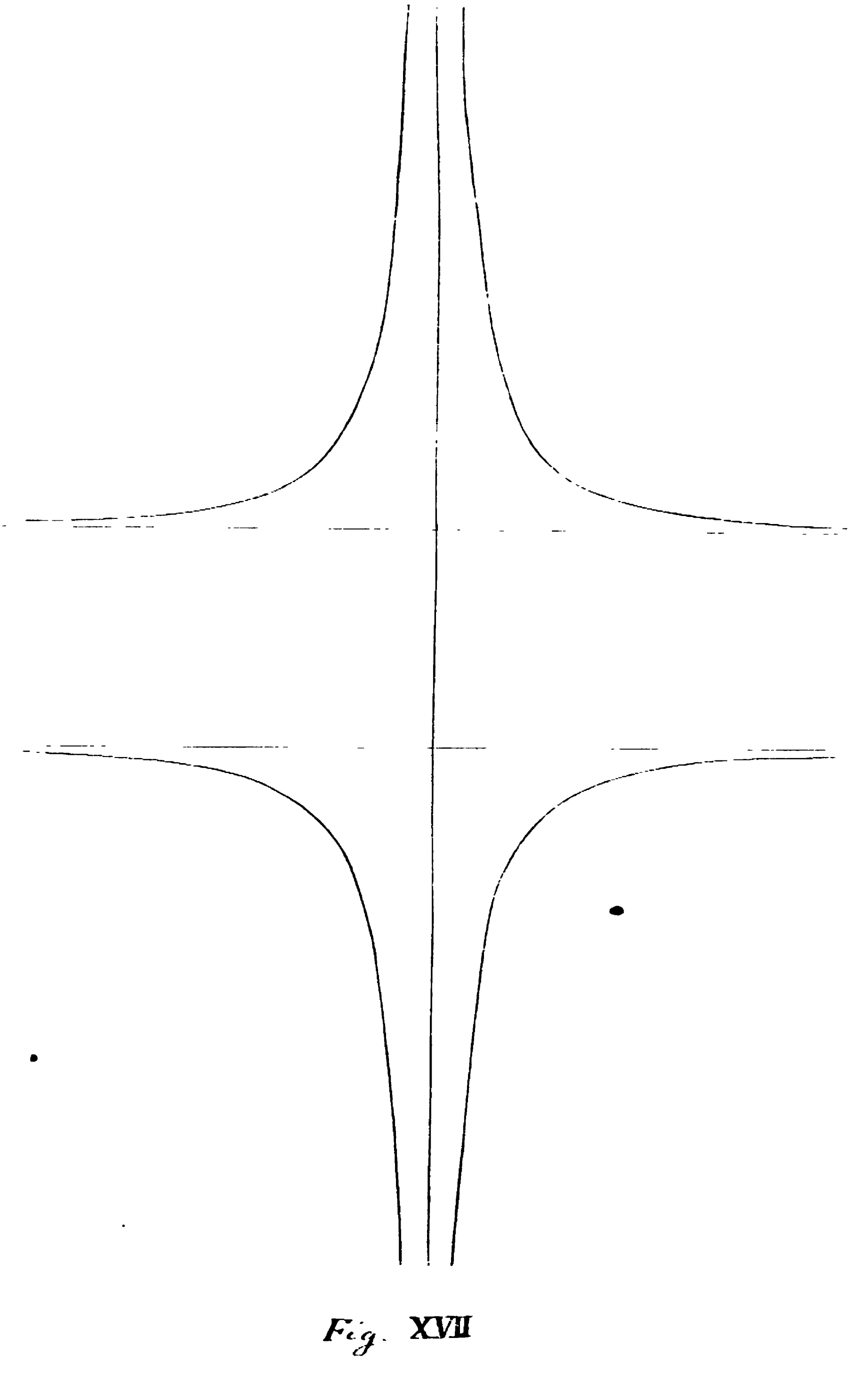


Fig. XVII

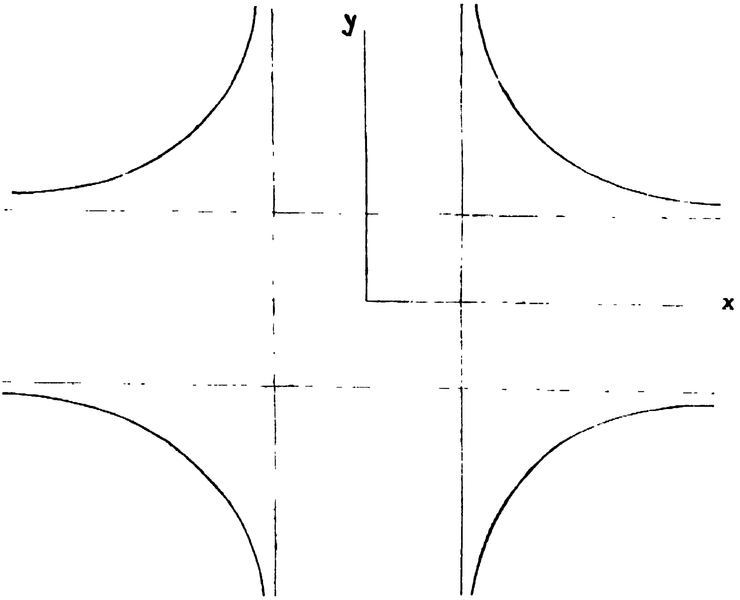


Fig. XVIII

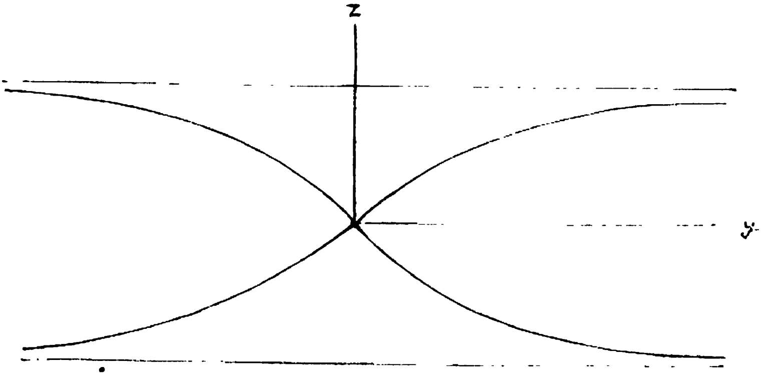


Fig. XIX.

ON HARMONICS ASSOCIATED WITH AN ELLIPSOID.*

SUDHANSUKUMAR BANERJI, D.Sc., *Sir Rashbehari Ghosh Professor of Applied Mathematics, University of Calcutta.*

PART I. THEORY..

1. *Introduction.*

In a celebrated memoir† published in the year 1835, Lamé transformed the equation $\nabla^2 V=0$, in the ellipsoidal coordinates (λ, μ, ν) representing a set of confocal ellipsoidal surfaces, hyperboloids of one sheet and hyperboloids of two sheets respectively and then broke it up into three ordinary linear differential equations in (λ, μ, ν). In a subsequent paper‡ the complete solutions of the equations were given in terms of a class of functions which are now commonly known as *Lamé's functions*. These functions have been developed in a series of articles by Heine, Liouville and a number of other investigators. An elaborate theory of ellipsoidal harmonics was subsequently given by Niven in a memoir published in the *Philosophical Transactions* of Royal Society in the year 1892. But although so much progress has made in the solution of Laplace's equation $\nabla^2 V=0$, the wave equation $(\nabla^2+k^2)\ V=0$ has hitherto resisted all our attempts in solving it in the coordinates (λ, μ, ν). The wave equation was first transformed by Mathieu§ in these coordinates, but the transformed equation was found to be so unmanageable that he had to content himself with approximating to its solution for the special case of an ellipsoid of revolution. Subsequent writers including Prof. Niven|| have simply improved upon this approximation of Mathieu.

In the present paper a class of harmonics which are solutions of

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* Some preliminary results on this subject have been published in the *Bulletin of the Calcutta Mathematical Society*, Vol. X, Nos. 2 and 3.

† *Memoires des Savants Étrangers*, Vol. V. Although the volume is dated 1838, this paper (which was reprinted in *Liouvilles Journal*, 1837) must have appeared at least as early as 1835.

‡ *Liouville's Journal*, 1839.

§ *Cours de Physique Mathématique*, Ch. IX.

|| *Phil, Trans.*, Vol. CLXXI (1880).

Laplace's equation $\nabla^2 V=0$ have been worked out in the ellipsoidal coordinates (ρ, θ, ϕ) defined by

$$\left.\begin{aligned} x &= a\rho \sin\theta\cos\phi, \\ y &= b\rho \sin\vartheta\sin\phi, \\ z &= c\rho\cos\theta, \end{aligned}\right\} \qquad (1)$$

where ρ=constant obviously determines a set of similar and similarly situated ellipsoids. This system of coordinates being analogous to the ordinary polar coordinates has got certain advantages over the system (λ, μ, ν), but also labours under certain disadvantages in as much as it does not form an orthogonal system. The harmonics in the coordinates (ρ, θ, ϕ) developed in this paper will be found to bear a close resemblance in forms as well as many of their principal properties to the tesseral harmonics. Owing to this analogy most of the methods which are now in common use for the treatment of spherical problems can be easily extended to solve similar problems for an ellipsoidal boundary. These harmonics will also be found to be simpler and more convenient for application to physical problems than the Lamé's functions.

A difficulty is however experienced when an attempt is made to construct a set of elementary solutions of the wave equations $(\nabla^2+k^2)\, V=0$ in the coordinates (ρ, ϑ, ϕ), in view of the fact that the transformed equation in the coordinates (ρ, ϑ, ϕ) is not separable into differential equations involving either ρ only or ϑ and ϕ only. This is what one should expect, for long ago Weber* remarked that the elliptic and the parabolic substitutions are the only transformations which lead to elementary solutions of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+k^2 V=0.$$

The same should obviously be true of the three-dimensional equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+k^2 V=0,$$

namely, that the ellipsoidal (λ, μ, ν) and the paraboloidal substitutions are the only transformations which should lead to elementary solutions of this equation.

But although it is not possible to construct a set of rigorous elementary solutions of the wave equation in the coordinates (ρ, θ, ϕ), a class of approximate solutions can be easily obtained. It is thought that as we are not yet in possession of any satisfactory solution of this equation in the system (λ, μ, ν), the approximate solutions here given in the coordinates (ρ, θ, ϕ) may perhaps be used with advantage to elucidate some of the obscure points in the ellipsoidal problems. With this object in view a few typical applications of these solutions will be considered towards the end of this paper.

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* *Math. Ann.*, Bd. 1. See also Häntzschel, *Reduction der Potentialgleichung* p. 137.

2. *Transformation and solution of the equation* $\nabla^2 V = 0$ *in the coordinates* (ρ, θ, ϕ).

If the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

be transformed in the coordinates (ρ, θ, ϕ) by the usual method, we obtain

$$\begin{aligned}
&\frac{\partial^2 V}{\partial \rho^2}\left[\sin^2\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) + \frac{\cos^2\theta}{c^2}\right] + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}\left[\cos^2\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) + \frac{\sin^2\theta}{c^2}\right] \\
&+ \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) + \frac{2}{\rho}\frac{\partial^2 V}{\partial \rho \partial \theta}\left[\sin\theta\cos\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2} - \frac{1}{c^2}\right)\right] \\
&\quad + \frac{2}{\rho}\frac{\partial^2 V}{\partial \rho \partial \phi}\sin\phi\cos\phi\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{2}{\rho^2}\frac{\partial^2 V}{\partial \theta \partial \phi}\sin\phi\cos\phi\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cot\theta \\
&\quad + \frac{1}{\rho}\frac{\partial V}{\partial \rho}\left[\cos^2\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) + \frac{\sin^2\theta}{c^2} + \left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right)\right] \\
&+ \frac{1}{\rho^2}\frac{\partial V}{\partial \theta}\left[\cot\theta\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) - 2\sin\theta\cos\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2} - \frac{1}{c^2}\right)\right] \\
&\qquad + \frac{2}{\rho^2}\frac{\partial V}{\partial \phi}\operatorname{cosec}^2\theta\cos\phi\sin\phi\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0. \qquad (2)
\end{aligned}$$

If it is assumed that this equation has a solution of the form

$$V = \rho^n U_n, \qquad (3)$$

n being a positive or a negative integer and U_n a function of θ and ϕ only, then U_n satisfies the equation

$$\begin{aligned}
&\frac{\partial^2 U_n}{\partial \theta^2}\left[\cos^2\theta\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) + \frac{\sin^2\theta}{c^2}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 U_n}{\partial \phi^2}\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) \\
&+ 2\frac{\partial^2 U_n}{\partial \theta \partial \phi}\sin\phi\cos\phi\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cot\theta + \frac{\partial U_n}{\partial \theta}\left[2n\sin\theta\cos\theta\left(\frac{\cos^2\phi}{a^2} + \right.\right.
\end{aligned}$$

$$\frac{\sin^2\phi}{b^2}-\frac{1}{c^2}\bigg)+\cot\theta\left(\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{b^2}\right)-2\sin\theta\cos\theta\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}-\frac{1}{c^2}\right)\Bigg]$$

$$+\frac{\partial U_n}{\partial\phi}\left[2n\sin\phi\cos\phi\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+2\operatorname{cosec}^2\theta\sin\phi\cos\phi\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right]$$

$$+n(n-1)\left[\sin^2\theta\left(\frac{\cos^2\phi}{a^4}+\frac{\sin^2\phi}{b^2}\right)+\frac{\cos^2\theta}{c^2}\right]U_n+n\Bigg[\cos^2\theta\bigg(\frac{\cos^2\phi}{a^2}+$$

$$\frac{\sin^2\phi}{b^2}\bigg)+\frac{\sin^2\theta}{c^2}+\left(\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{b^2}\right)\Bigg]U_n=0. \qquad (4)$$

It is not easy to determine the form of the function U_n from this differential equation. The method described in the next article however leads to the form of the function without much difficulty.

3. *Internal harmonics in the coordinates* $(\rho,\ \theta,\ \phi)$.

It is well known that if $(r,\ \theta',\ \phi')$ denote the spherical polar coordinates of a point $(x,\ y,\ z)$, then

$$r^nP_n^m(\cos\theta')\begin{matrix}\cos\\ \sin\end{matrix}m\phi'=\frac{(n+m)\,!}{2\pi n\,!\ i^m}\int_{-\pi}^{\pi}(z+ix\cos u+iy\sin u)^n\begin{matrix}\cos\\ \sin\end{matrix}mudu. \qquad (5)$$

Obviously by a generalisation of this expression we can define a function $C_n^m(\theta,\ \phi)$ by the expression

$$C_n^m(\theta,\ \phi)=\frac{(n+m)\,!}{2\pi n\,!\ i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^n\cos mudu. \qquad (7)$$

With this definition for the function $C_n^m(\theta,\ \phi)$, it is easy to see that $\rho^n C_n^m(\theta,\ \phi)$ is a solution of Laplace's equation in the coordinates $(\rho,\ \theta,\ \phi)$ defined by (1)

Similarly we can define a function $S_n^m(\theta,\ \phi)$ by the expression

$$S_n^m(\theta,\ \phi)=\frac{(n+m)\,!}{2\pi n\,!\ i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\quad\sin\phi\sin u)^n.$$
$$\sin mudu. \qquad (8)$$

and $\rho^n S_n^m(\theta,\ \phi)$ is another solution of Laplace's equation in the coordinates $(\rho,\ \theta,\ \phi)$.

One form of the function U_n defined in the previous article is $C_n^m(\theta,\ \phi)$ and another is $S_n^m(\theta,\ \phi)$. Both these functions therefore satisfy the differential equation (4).

The function corresponding to the zonal harmonics can be defined by

$$C^0_n(\theta, \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (c \cos\theta + ia \sin\theta \cos\phi \cos u + ib \sin\theta \sin\phi \sin u)^n \, du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [c \cos\theta + i\,(a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi)^{\frac{1}{2}} \cos u]^n \, du. \tag{9}$$

When $a=b$, that is, for an ellipsoid of revolution, the function becomes independent of ϕ and reduces to

$$C^0_n(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (c \cos\theta + ia \sin\theta \cos u)^n \, du. \tag{10}$$

For convenience we shall usually write $C^0_n(\theta)$ as $C_n(\theta)$.

For elucidating some of the properties of these functions, it will be convenient sometimes to denote the functions $C^m_n(\theta, \phi)$ and $S^m_n(\theta, \phi)$ defined as above by the simpler notations $C^m_n(c \cos\theta)$ and $S^m_n(c \cos\theta)$ always however remembering that these are functions of both θ and ϕ

Since x, y, z can be interchanged in the cyclical order in the expression (5), it is obvious that the function $C^m_n(\theta, \phi)$ can have the two other forms given below :—

$$\frac{(n+m)!}{2\pi n!\; i^m}\int_{-\pi}^{\pi} (a \sin\theta \cos\phi + ic \cos\theta \cos u + ib \sin\theta \sin\phi \sin u)^n \cos mu\,du, \tag{11}$$

$$\frac{(n+m)!}{2\pi n!\; i^m}\int_{-\pi}^{\pi} (b \sin\theta \sin\phi + ia \sin\theta \cos\phi \cos u + ic \cos\theta \sin u)^n \cos mu\,du. \tag{12}$$

To distinguish these two expressions from (7) we denote them by $C^m_n(a \sin\theta \cos\phi)$ and $C^m_n(b \sin\theta \sin\phi)$ respectively.

Similarly the function $S^m_n(\vartheta, \phi)$ can have the two following different forms :—

$$\frac{(n+m)!}{2\pi n!\; i^m}\int_{-\pi}^{\pi} (a \sin\theta \cos\phi + ic \cos\theta \cos u + ib \sin\theta \sin u)^n \sin mu\,du, \tag{13}$$

$$\frac{(n+m)\,!}{2\pi n\,!\;i^m}\int_{-\pi}^{\pi}(b \sin\theta \sin\phi + ia \sin\theta \cos\phi \cos u + ic \cos\theta \sin u)^n \sin mu du. \quad (14)$$

These two expressions we shall denote by the notation S_n^m $(a \sin\theta \cos\phi)$ and S_n^m $(b \sin\theta \sin\phi)$, respectively.

The functions corresponding to the zonal harmonics, namely, the functions of the type C_n^0 (θ, ϕ) defined by (9) can be similarly denoted by C_n $(c \cos\theta)$, C_r $(a \sin\theta \cos\phi)$ and C_n $(b \sin\theta \sin\phi)$. The values of the functions C_n^m (θ, ϕ) and S_n^m (θ, ϕ) can also be expressed in terms of the hypergeometric function. Thus we get

$$C_n^m(c \cos\theta) = \frac{(n+m)\,!}{(n-m)\,!}\,\frac{i^m R^{2n+1} r^m \tan^m\theta}{2^m m\,!\;(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}, m+1,\right.$$
$$\left. -r^2 \tan^2\theta\right) \cos m\psi, \quad (15)$$

$$S_n^m(c \cos\theta) = \frac{(n+m)\,!\;i^m R^{2n+1} r^m \tan^m\theta}{(n-m)\,!\;2^m m\,!\;(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}, m+1,\right. \quad (16)$$
$$\left. -r^2 \tan^2\theta\right) \sin m\psi,$$

where F is a hypergeometric function of the four elements within the parenthesis and

$$\left.\begin{aligned} R^2 &= (a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta), \\ r^2 &= (a^2 \cos^2\phi + b^2 \sin^2\phi)/c^2, \\ \tan\psi &= \frac{b}{a} \tan\phi, \end{aligned}\right\} \quad (17)$$

c being assumed to be the greatest axis of the ellipsoid.

1. *External harmonics in the coordinates* (ρ, θ, ϕ).

To obtain the harmonics which vanish at infinity we define the functions

$$\mathfrak{C}_n^m(\theta, \phi) = (-1)^m \frac{n(n-1)\,.\,.\,.\,(n-m+1)}{2\pi}$$

$$\int^{2\pi} \frac{\cos mu du}{(c \cos\theta + ia \sin\theta \cos\phi \cos u + ib \sin\theta \sin\phi \sin u)^{n+1}}, \quad (18)$$

and $\mathfrak{S}_r^m(\theta, \phi) = (-1)^m \dfrac{n\,(n-1)\,.\,.\,.\,(n-m+1)}{2\pi}$

$$\int_0^{2\pi} \frac{\sin mu du}{(c \cos \theta + ia \sin \theta \cos \phi \cos u + ib \sin \theta \sin \phi \sin u)^{n-1}} \qquad (19)$$

and obviously $\mathfrak{C}_n^m (\theta, \phi)/\rho^{n+1}$ and $\mathfrak{S}_n^m (\theta, \phi)/\rho^{n+1}$ are solutions of Laplace's equation which vanish at infinity.

The function $\mathfrak{C}_n^m (\theta, \phi)$ can have also the following two different forms:—

$$(-1)^m \frac{n(n-1)\ldots(n-m+1)}{2\pi}$$

$$\int_0^{2\pi} \frac{\cos mu du}{(a \sin \theta \cos \phi + ic \cos \theta \cos u + ib \sin \theta \sin \phi \sin u)^{n+1}}, \qquad (20)$$

and
$$(-1)^m \frac{n(n-1)\ldots(n-m+1)}{2\pi}$$

$$\int_0^{2\pi} \frac{\cos mu du}{(b \sin \theta \sin \phi + ia \sin \theta \cos \phi \cos u + ic \cos \theta \sin u)^{n+1}}. \qquad (21)$$

and the function $\mathfrak{S}_n^m (\theta, \phi)$ the forms:—

$$(-1)^m \frac{n(n-1)\ldots(n-m+1)}{2\pi}$$

$$\int_0^{2\pi} \frac{\sin mu du}{a \sin \theta \cos \phi + ic \cos \theta \cos u + ib \sin \theta \sin \phi \sin u)^{n+1}}, \qquad (22)$$

$$(-1)^m \frac{n(n-1)\ldots(n-m+1)}{2\pi}$$

$$\int_0^{2\pi} \frac{\sin mu du}{(b \sin \theta \sin \phi + ia \sin \theta \cos u + ic \cos \theta \sin u)^{n+1}} \qquad (23)$$

As in the case of the internal harmonics, we denote the three different forms of $\mathfrak{C}_n^m (\theta, \phi)$ and $\mathfrak{S}_n^m (\theta, \phi)$ by the simpler notations $\mathfrak{C}_n^m (c \cos \theta)$, $\mathfrak{C}_n^m (a \sin \theta \cos \phi)$, $\mathfrak{C}_n^m (b \sin \theta \sin \phi)$ and $\mathfrak{S}_n^m (c \cos \theta)$, $\mathfrak{S}_n^m (a \sin \theta \cos \phi)$, $\mathfrak{S}_n^m (b \sin \theta \sin \phi)$ respectively.

The functions corresponding to the zonal harmonics, namely, the three functions of the type

$$\frac{1}{2\pi}\int_{0}^{2\pi}\frac{du}{(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^{n+1}} \qquad (24)$$

can be similarly denoted by $\mathfrak{C}_{n}^{m}(c\cos\theta)$, $\mathfrak{C}_{n}(a\sin\theta\cos\phi)$ and $\mathfrak{C}_{n}(b\sin\theta\sin\phi)$ respectively.

The functions $C_{n}^{m}(\theta,\phi)$, $S_{n}^{m}(\theta,\phi)$, $\mathfrak{C}_{n}^{m}(\theta,\phi)$ and $\mathfrak{S}_{n}^{m}(\theta,\phi)$ which we have so far defined represent four different forms of the function U_{n} and they all satisfy the differential equation (4).

It is interesting to note the following relations between the functions $C_{n}^{m}(\theta,\phi)$, $S_{n}^{m}(\theta,\phi)$, $\mathfrak{C}_{n}^{m}(\theta,\phi)$ and $\mathfrak{S}_{n}^{m}(\theta,\phi)$:—

$$\left.\begin{aligned} C_{n}^{m}(\theta,\phi)&=(a^{2}\sin^{2}\theta\cos^{2}\phi+b^{2}\sin^{2}\theta\sin^{2}\phi+c^{2}\cos^{2}\theta)^{\frac{2n+1}{2}}\,\mathfrak{C}_{n}^{m}(\theta,\phi),\\ S_{n}^{m}(\theta,\phi)&=(a^{2}\sin^{2}\theta\cos^{2}\phi+b^{2}\sin^{2}\theta\sin^{2}\phi+c^{2}\cos^{2}\theta)^{\frac{2n+1}{2}}\,\mathfrak{S}_{n}^{m}(\theta,\phi).\end{aligned}\right\} \qquad (25)$$

When $a=b=c=1$,

$$\left.\begin{aligned} C_{n}^{m}(\theta,\phi)&=\mathfrak{C}_{n}^{m}(\theta,\phi)=P_{n}^{m}(\cos\theta)\cos m\phi,\\ \text{and } S_{n}^{m}(\theta,\phi)&=\mathfrak{S}_{n}^{m}(\theta,\phi)=P_{n}^{m}(\cos\theta)\sin m\phi.\end{aligned}\right\} \qquad (26)$$

5. *Functions corresponding to the Laplacians.*

It is well known that

$$P_{n}(\sin u\cos v\sin u'\cos v'+\sin u\sin v\sin u'\sin v'+\cos u\cos u')$$
$$=P_{n}(\cos u)\,P_{n}(\cos u')+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\,P_{n}^{m}(\cos u)\,P_{n}^{m}(\cos u')\cos m(v-v'). \qquad (27)$$

If we write

$$x=r\sin u\cos v \qquad x'=r'\sin u'\cos v',$$
$$y=r\sin u\sin v, \qquad y'=r'\sin u'\sin v',$$
$$z=r\cos u, \qquad z'=r'\cos u',$$

then the above identity can be written in the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[xx'+yy'+zz'+i\{(yz'-zy')^{2}+(zx'-xz')^{2}+(xy'-yx')^{2}\}^{\frac{1}{2}}\cos w]\,du$$

$$= \frac{1}{4\pi^2}\int_{-\pi} (z + ix \cos w + iy \sin w)^n \, dw \times \int_{-\pi}^{\pi} (z' + ix \cos w + iy' \sin w)^n \, dw$$

$$+ 2 \sum_{m=1}^{m=n} \frac{(n-m)! \; (n+m)!}{4\pi^2 (n!)^2} (-1)^m \left[\int_{-\pi}^{\pi} (z + ix \cos w + iy \sin w)^n \cos mw dw \right.$$

$$\times \int_{-\pi}^{\pi} (z + ix' \cos w + iy' \sin w)^n \sin mw dw$$

$$\left. + \int_{-\pi}^{\pi} (z + ix \cos w + iy \sin w)^n \sin mw dw \times \int_{-\pi}^{\pi} (z' + ix' \cos w + iy' \sin w)^n \sin mw dw \right] \tag{28}$$

Now if we write

$$x' = a \rho \sin \theta \cos \phi, \qquad x = r \sin u \cos v,$$
$$y' = b \rho \sin \theta \sin \phi, \qquad y = r \sin u \sin v,$$
$$z' = c \rho \cos \theta, \qquad z = r \cos u,$$

we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [a \sin \theta \cos \phi \sin u \cos v + b \sin \theta \sin \phi \sin u \sin v + c \cos \theta \cos u$$

$$+ i \{ b \sin \theta \cos \phi \cos u - c \sin u \sin v \cos \theta)^2 + (c \sin u \cos v \cos \theta - a \sin \theta \cos \phi$$
$$\cos u)^2 + (a \sin \theta \cos \phi \sin u \sin v - b \sin \theta \sin \phi \sin u \cos v)^2\}^{\frac{1}{2}} \cos w] dw$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} (c \cos \theta + ia \sin \theta \cos \phi \cos w + ib \sin \theta \sin \phi \sin w)^n \, dw$$

$$\int_{-\pi}^{\pi} (\cos u + i \sin u \cos v \cos w + i \sin u \sin v \sin w)^n \, dw$$

$$+ 2 \sum_{m=1}^{m=n} \frac{(n-m)! \; (m+n)!}{4\pi^2 (n!)^2} (-1)^m$$

$$\left[\int_{-\pi}^{\pi} (c \cos \theta + ia \sin \theta \cos \phi \cos w + ib \sin \theta \sin \phi \sin w)^n \cos mw dw \right.$$

$$\times \int_{-\pi}^{\pi} (\cos u + i \sin u \cos v \cos w + i \sin u \sin v \sin w)^n \cos mw dw$$

$$+ \int_{-\pi}^{\pi} (c \cos\theta + ia \sin\theta \cos\phi \cos w + ib \sin\theta \sin\phi \sin w)^n \sin mw dw$$

$$\times \int_{-\pi}^{\pi} (\cos u + i \sin u \cos v \cos w + i \sin u \sin v \sin w)^n \sin mw dw \Bigg]. \quad (29)$$

In other words we get in accordance with our previous definitions,

$$C_n\,(a \sin\theta \cos\phi \sin u \cos v + b \sin\theta \sin\phi \sin u \sin v + c \cos\theta \cos u)$$

$$= C_n\,(c \cos\theta)\, P_n(\cos u) + 2 \sum_{m=1}^{m=n} \frac{(n-m)\,!}{(n+m)\,!} \Big[C_n^m\,(c \cos\theta)\, P_n^m\,(\cos u) \cos mv$$

$$+ S_n^m\,(c \cos\theta)\; P_n^m\,(\cos u) \sin mv \Big]. \quad (30)$$

By a similar process, it is easy to show that

$$C_n\,(a \sin\theta \cos\phi \sin u \cos v + b \sin\theta \sin\phi \sin u \sin v + c \cos\theta \cos u)$$

$$= C_n\,(c \cos u)\, P_n(\cos\theta) + 2 \sum_{m=1}^{n} \frac{(n-m)\,!}{(n+m)\,!} \Big[C_n^m(c \cos u)\; P_n^m\,(\cos\theta) \cos m\,\phi$$

$$+ S_n^m\,(c \cos u)\, P_n^m\,(\cos\theta) \sin m\,\phi \Big]. \quad (31)$$

In a similar way, we obtain

$$C_n\,(a \sin\theta \cos\phi.\ \alpha \sin u \cos v + b \sin\theta \sin\phi.\ \beta \sin u \sin v + c \cos\theta.\ \gamma \cos u)$$

$$= C_n\,(c \cos\theta)\, C_n\,(\gamma \cos u) + 2 \sum_{m=1}^{m=n} \frac{(n-m)\,!}{(n+m)\,!} \Big[C_n^m\,(c \cos\theta)\;\; C_n^m(\gamma \cos u)$$

$$+ S_n^m\,(c \cos\theta)\, S_n^m\,(\gamma \cos u) \Big]. \quad (32)$$

on making the substitutions

$x = a\rho \sin\theta \cos\phi$, $y = b\,\rho \sin\theta \sin\phi$, $z = c\,\rho \cos\theta$,

$x' = \alpha\,\rho' \sin u \cos v$. $y' = \beta\,\rho' \sin u \sin v$, $z = \gamma\,\rho' \cos u$.

These functions correspond to the Laplacians of the nth degree.

6. *Conjugate Properties satisfied by the harmonics.*

By an application of Green's theorem we can prove that the functions $C_n^m\,(\theta, \phi)$, $S_n^m\,(\theta, \phi)$. $\mathfrak{C}_n^m\,(\theta, \phi)$ and $\mathfrak{S}_n^m\,(\theta', \phi)$ all satisfy certain conjugate properties.

The element of volume in the coordinates (ρ, θ, ϕ) is

$$abc\, \rho^2 \sin\theta\, d\rho d\theta d\phi, \quad (33)$$

which can also be written in the form $dp.dS$, where dS is a surface element on the ellipsoid and dp an element of perpendicular on the surface element. The element of normal to the surface ρ=constant, defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\rho^2. \tag{34}$$

is given by

$$dn=\frac{1}{h}\left(\frac{\partial\rho^2}{\partial x}dx+\frac{\partial\rho^2}{\partial y}dy+\frac{\partial\rho^2}{\partial z}dz\right)=\frac{d\rho^2}{h}=\frac{2\rho d\rho}{h}, \tag{35}$$

where
$$\begin{aligned} h^2&=\left(\frac{\partial\rho^2}{\partial x}\right)^2+\left(\frac{\partial\rho^2}{\partial y}\right)^2+\left(\frac{\partial\rho^2}{\partial z}\right)^2\\ &=4\rho^2\left[\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2}\right]\\ &=\frac{4\rho^2}{p_0^2}\ \text{(say)}, \end{aligned}$$

where p_0 is the length of the perpendicular on the ellipsoidal surface $\rho=1$ of semi-axes (a, b, c).

Therefore $dn=dp=p_0 d\rho$, and

$$dS=\frac{abc}{p_0}\rho^2\sin\theta\,d\theta d\phi. \tag{36}$$

Now, by Green's theorem, if Φ and Φ' be two functions which satisfy Laplace's equation, we have

$$\iint\left(\Phi'\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial\Phi'}{\partial n}\right)dS=0.$$

Hence since $\frac{\partial}{\partial n}=\frac{1}{p_0}\frac{\partial}{\partial\rho}$, we see at once that the functions $C_n^m(\theta, \phi)$, $S_n^m(\theta, \phi)$, $\mathfrak{C}_n^m(\theta, \phi)$, $\mathfrak{S}_n^m(\theta, \phi)$ must satisfy the following conjugate properties :—

$$\int_0^\pi\int_0^{2\pi} U_n U_{n'}\frac{\sin\theta}{p_0^2}d\theta d\phi=0\,(n\neq n'), \tag{37}$$

and
$$\int_0^\pi\int_0^{2\pi}[U_n]^2\frac{\sin\theta}{p_0^2}d\theta d\phi=\text{constant}=\lambda_n\ \text{(say)}, \tag{38}$$

where U_n stands for $C_n^m(\theta, \phi)$, $S_n^m(\theta, \phi)$, $\mathfrak{C}_n^m(\theta, \phi)$ or $\mathfrak{S}_n^m(\theta, \phi)$.

To obtain the value of the integral (38), we write $\xi = \sin\theta\cos\phi$, $\eta = \sin\theta\sin\phi$, $\zeta = \cos\theta$, then since U_n is a homogeneous function of degree n in (ξ, η, ζ) and $\frac{1}{p_0^2}$ a homogeneous function of degree 2 in (ξ, η, ζ), we can write the integral (38) in the form

$$\iint \Big[U_n(\xi, \eta, \zeta) \Big]^2 \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right) dS \qquad (39)$$

where the integration is carried over the surface of a unit sphere. The value or this integral can be expressed in a variety of forms. One way* of expressing the value of the integral is

$$\frac{4\pi\,(n+1)!}{(2n+3)!} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)_b \Sigma\, 2^q\, U_2(\xi, \eta, \zeta),$$

where $U_q(\xi, \eta, \zeta) = \frac{1}{[\frac{1}{2}(2n-q)]!} \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} \right)^{\frac{1}{2}(2n-q)} [U_n(\xi, \eta, \zeta])^2$, (40)

and q has all the finite number of positive integral values for which $\frac{1}{2}(2n-q)$ is integral and not negative.

We can thus write

$$\int_0^{\pi}\!\!\int_0^{2\pi} \Big[C_n^m(\theta, \phi) \Big]^2 \frac{\sin\theta}{p_0^2}\, d\theta d\phi$$

$$= \frac{[(n+m)!]^2\,(n+1)}{4\pi n!} (-1)^n \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \sum_{q=0}^{q-n} 2^q \frac{1}{(n-q)!}$$

$$\cdot \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} \right)^{n \quad q2} \left[\int_{-\pi}^{\pi} (\zeta + i\xi\cos u + i\,\eta\sin u)^n \cos mu\,du \right]$$

(41)

Similarly for the functions $S_n^m(\theta, \phi)$ $\mathfrak{C}_n^m(\theta, \phi)$ and $\mathfrak{S}_n^m(\theta, \phi)$.

Owing to these conjugate properties an arbitrary function of (θ, ϕ) can obviously be expanded in a series of these functions, the conditions of expansibility being the same as those which govern a Fourier expansion or an expansion in a series of tesseral harmonics.

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* See a note by Prof. H. F. Baker on a formula connected with the theory of sperical harmonics, *Proc. Lond. Math. Soc.*, Vol. XV (1916) and a note by Dr. Bromwhich in the same number.

7. *Potential problems involving ellipsoidal boundaries.*

A number of interesting potential problems can be solved with the help of the functions whose properties we have studied in the preceding articles.

The familiar method of determining by means of ordinary spherical harmonics the potential of a spherical bowl or a circular disc at any arbitary point from the known value of the potential on the axis becomes at once available for solving similar problems for an ellipsoidal bowl or an elliptic plate. For example, assuming the bowl to form part of the ellipsoidal surface $\rho=1$ and its axes to coincide with the c-axis of the ellipsoid, we easily obtain the potential of the bowl at any point on the axis as a function of ρ, say $f(\rho)$. Then for points for which $\rho \angle 1$, we expand $f(\rho)$ in the form

$$f(\rho)=a_0+a_1\rho+a_2\rho^2+\dots ,$$

and for points for which $\rho > 1$, we expand $f(\rho)$ in the form

$$f(\rho)=\frac{b_0}{\rho}+\frac{b_1}{\rho^2}+\frac{b_2}{\rho^3}+\dots .$$

Now since the function $C_n(c\cos\theta)$ has the value c^n on the axis $\theta=0$, the potential of the bowl at points for which $\rho \angle 1$, is given by

$$V=a_0+\frac{a_1}{c}\rho\, C_1(c\cos\theta)+\frac{a_2}{c^2}\rho^2 C_2(c\cos\theta)+\frac{a_3}{c^3}\rho^3 C_3(c\cos\theta)+\dots$$

and since the function $\mathfrak{C}_n(c\cos\theta)$ has the value $\frac{1}{c^{n-1}}$ on the axis, the potential at points for which $\rho > 1$, is given by

$$V=\frac{b_0 c}{\rho}\mathfrak{C}_0(c\cos\theta)+\frac{b_1c^2}{\rho^2}\mathfrak{C}_1(c\cos\theta)+\frac{b_2c^3}{\rho^3}\mathfrak{C}_2(c\cos\theta)+\dots$$

When the surface distribution on the ellipsoid $\rho=1$ is given as a function of $f(\theta,\phi)$, we expand $f(\theta,\phi)$ in a series of the type when $\rho \angle 1$,

$$f(\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{n,m}\,C_n^m(\theta,\phi)$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{n}B_{r,m}\,S_n^m,(\theta,\phi)$$

and the potential at points for which $\rho \angle 1$, is given by

$$V=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{n,m}\,\rho^n\,C_n^m(\theta,\phi)$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{n,m}\,\rho^n\,S_n^m(\theta,\phi)$$

For points for which $\rho > 1$, we expand $f(\theta,\phi)$ in a series of the type

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n,m} \quad \mathfrak{C}_n^m(\theta, \phi)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{n} \beta_{n,m} \quad \mathfrak{S}_n^m(\theta, \phi),$$

and then the potential at points for which $\rho > 1$, is given by

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n,m} \quad \mathfrak{C}_n^m(\theta, \phi)/\rho^{n+1}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{n} \beta_{n,m} \quad \mathfrak{S}_n^m(\theta, \phi)/\rho^{n+1}$$

8. *Two dimensional harmonics.*

To obtain the two dimensional harmonics in the coordinates (ρ, θ) defined by $x = a\rho\cos\theta$, $y = b\rho\sin\theta$, we notice that since $(x+iy)^n$ is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

that $\rho^n(a\cos\theta + i\,b\sin\theta)^n$ is a solution of this equation. Now putting

$$a\cos\theta = R\cos\psi, \; b\sin\theta = R\sin\psi,$$

where $$R = (a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}, \; \tan\psi = \frac{b}{a}\tan\theta,$$

we get the two dimensional harmonics in the forms

$$\rho^n R^n \cos n\psi,$$

$$\rho^n R^n \cos n\psi.$$

Hence the complete set of two-dimensional harmonics are

$$\log \rho R, \; \rho R\cos\psi, \; \rho^2 R^2 \cos 2\psi, \; \rho^3 R^3 \cos 3\psi, \; \ldots$$

$$\rho R\sin\psi, \; \rho^2 R^2 \sin 2\psi, \; \rho^3 R^3 \sin 3\psi, \; \ldots,$$

9. *The wave equation in the coordinates* (ρ, θ, ϕ).

In this article we shall give a method of constructing a set of approximate solutions of the wave equation

$$(\nabla^2 + k^2)\, V = 0,$$

in the ellipsoidal coordinates (ρ, θ, ϕ). The possibility of the solution being expressed in the form of a product $\Psi_n(k\rho)\, C_n^m(\theta, \phi)$ is tacitly assumed in the method. It will be noticed, as we have said before, that this assumption is not rigorously justifiable, but when the assumption has been made, it is possible to obtain an expression for $\Psi_n(k\rho)$

which represents the mean value of the function on the ellipsoid ρ and that with this value for $\Psi_n\ (k\rho)$, the quantity $\Psi_n\ (k\rho)\ C_n^m\ (\theta, \phi)$ very nearly approaches a solution.

If we put $x=ax'$, $y=by'$, $z=cz'$, then the wave equation can be written in the form

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 V}{\partial y'}+\frac{1}{c^2}\frac{\partial^2 V}{\partial z'^2}+k^2V=0.$$

If it is assumed that the equation has a solution of the form

$$R_n\,U_n,$$

where R_n is a function of ρ only and U_n is a solution of the equation

$$\frac{1}{a^2}\frac{\partial^2 U_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 U_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial\ U_n}{\partial z'^2}=0,$$

then we obtain

$$\frac{1}{a^2}\frac{\partial^2(R_nU_n)}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2(R_nU_n)}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2(R_nU_n)}{\partial z'^2}+k^2R_nU_n=0,$$

that is,

$$\left[\frac{1}{a^2}\frac{\partial^2 R_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 R_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 R_n}{\partial z'^2}\right]+2\left[\frac{1}{a^2}\frac{\partial R_n}{\partial x'}\frac{\partial U_n}{\partial x'}\right.$$
$$\left.\frac{1}{b^2}\frac{\partial R_n}{\partial y'}\frac{\partial U_n}{\partial y'}+\frac{1}{c^2}\frac{\partial R_n}{\partial z'}\frac{\partial U_n}{\partial z'}\right]+k^2R_nU_n=0.$$

Now since R_n is a function of ρ only, we have

$$\frac{\partial R_n}{\partial x'}=\frac{x'}{\rho}\frac{\partial R_n}{\partial\rho},\quad \frac{\partial R_n}{\partial y'}=\frac{y'}{\rho}\frac{\partial R_n}{\partial\rho},\quad \frac{\partial R_n}{\partial z'}=\frac{z'}{\rho}\frac{\partial R_n}{\partial\rho}.$$

Therefore the above equation becomes

$$\left[\frac{1}{a^2}\frac{\partial}{\partial x'}\left(\frac{x}{\rho}\frac{\partial R_n}{\partial\rho}\right)+\frac{1}{b^2}\frac{\partial}{\partial y'}\left(\frac{y'}{\rho}\frac{\partial R_n}{\partial\rho}\right)+\frac{1}{c^2}\frac{\partial}{\partial z'}\left(\frac{z'}{\rho}\frac{\partial R_n}{\partial\rho}\right)\right]U_n$$
$$+\frac{2}{\rho}\frac{\partial R_n}{\partial\rho}\left[\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c}\frac{\partial U_n}{\partial z'}\right]+k^2R_n\,U_n=0.$$

This can also be written in the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)\left[\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}\right]U_n$$
$$+\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\left[2\left(\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c}\frac{\partial U_n}{\partial z'}\right)\right.$$
$$\left.+U_n\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c}\right)\right]+k^2R_n\,U_n=0.$$

It will appear from this equation that it is not possible to separate the differential equation for R_n. But it is easy to obtain the differential equation satisfied by the mean value of R_n on any ellipsoidal surface. If we put $U_n = \rho^n C_n(\theta, \phi)$, this becomes

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right)\frac{C_n(\theta, \phi)}{p_0^2} + \frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\left[2n\, D_n(\theta, \phi) \right.$$

$$\left. + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) C_n(\theta, \phi)\right] + k\, R_n\, C_n(\theta, \rho) = 0,$$

where

$$\frac{1}{p_0^2} = \frac{\sin^2\theta \cos^2\phi}{a^2} + \frac{\sin^2\theta \sin^2\phi}{b^2} + \frac{\cos^2\theta}{c^2},$$

and

$$D_n{}^m(\theta, \phi) = \frac{(n+m)!}{2\pi n!\, i^m}\int_{-\pi}^{\pi}(c\cos\theta + ia\sin\theta\cos\phi\cos u$$

$$+ ib\sin\theta\sin\phi\sin u)^{n-1}$$

$$\left(\frac{\cos\theta}{c} + \frac{i\sin\theta\cos\phi\cos u}{a} + \frac{i\sin\theta\sin\phi\sin u}{b}\right)du.$$

Multiplying this equation by the conjugate function $C_n(\theta, \phi)$ and integrating we obtain

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta, \phi)]^2\frac{\sin\theta}{p_0^2}\,d\theta d\phi$$

$$+ \frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\left[2n\int_0^{\pi}\int_0^{2\pi} D_n(\theta, \phi)\, C'_n(\theta, \phi)\sin\theta\, d\theta d\phi\right.$$

$$\left. + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta, \phi)]^2\sin\theta\, d\theta d\phi\right]$$

$$+ k^2 R_n\int_0^{\pi}\int_0^{2\pi}[C_n(\theta, \phi)]^2\sin\theta\, d\theta d\phi = 0.$$

Now it is easy to see that

$$(2n+3)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta, \phi)]^2\frac{\sin\theta}{p_0^2}\,d\theta d\phi$$

$$= 2n\int_0^{\pi}\int_0^{2\pi} D_n(\theta, \phi)\, C_n(\theta, \phi)\sin\theta\, d\theta d\phi$$

$$+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta\,d\theta d\phi.$$

Hence the differential equation for R_n reduces to the form

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)+(2n+3)\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)$$

$$k^2\frac{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]\sin\theta\,d\theta d\phi}{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\frac{\sin\theta}{p_0^2}d\theta d\phi}R_n=0.$$

If now we write

$$k^2\frac{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta\,d\theta d\phi}{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\frac{\sin\theta}{p_0^2}d\theta d\phi}=k'^2,$$

the equation can be written in the form

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial}\right)+(2n+3)\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}+k'^2R_n=0,$$

which has the well-known solution

$$R_n=A\psi_n(k'\rho)+B\Psi_n(k'\rho),$$

where A and B are two arbitrary constants and

$$\psi_n(\rho)=\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\sin\rho}{\rho},\quad\Psi_n(\rho)=\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\cos\rho}{\rho}.$$

Also

$$\rho^n\psi_n(\rho)=\sqrt{\frac{\pi}{2\rho}}\,J_{n+\frac{1}{2}}(\rho),$$

$$\rho^n\Psi_n(\rho)=\sqrt{\frac{\pi}{2\rho}}\,J_{-n-\frac{1}{2}}(\rho).$$

The condition, which any two distinct solutions V, V' of the wave equation, which themselves or their differential coefficients in the direction of the normal vanish on the surface of the ellipsoid, must satisfy, namely

$$\int_0^1 \int_0^\pi \int_0^{2\pi} VV'\rho^2 \sin\theta \, d\rho d\theta d\phi = 0,$$

is also easily seen to be satisfied by these approximate solutions.

PART II. APPLICATIONS.

1. *Potential Problems*

As an example of the applications of the harmonics whose properties we have investigated in Part I, we shall here discuss the gravitational potential of a homogeneous semi-ellipsoid.

It is well known that the potential function due to a homogeneous hemisphere whose axis is taken as the polar axis is

$$V = \frac{M}{a}\left[\frac{a}{r} + \frac{3.1}{2.4}\frac{a^2}{r^2}P_1(\cos u) - \frac{3.1.1}{2.4.6}\frac{a^4}{r^4}P_3(\cos u) + \frac{3.1.1.3}{2.4.6.8}\frac{a^6}{r^8}P_5(\cos u) - \ldots\right],$$

if $r > a$, and is

$$V = \frac{M}{a}\left[\frac{3}{2} + \frac{3}{2}\frac{r}{a}P_1(\cos u) + \frac{r^2}{a^2}P_2(\cos u) + \frac{3.1}{2.4}\frac{r^3}{a^3}P_3(\cos u) \right.$$
$$\left. - \frac{3.1.1}{2.4.6}\frac{r^5}{a^5}P_5(\cos u) + \ldots\right]$$

if $r < a$ and $u > \frac{\pi}{2}$.

The potential of a homogeneous semi-ellipsoid can similarly be written in the form

$$V = M\left[\frac{A}{\rho}D + \frac{3.1}{2.4}\frac{A_1}{\rho}\mathfrak{C}_1^0(\theta, \phi) - \frac{3.1\,1}{2.4.6}\frac{A_3}{\rho^4}\mathfrak{C}_3^0(\theta, \phi) + \frac{3.1.1\,3}{2.4.6.8}\frac{A_5}{\rho^6}\mathfrak{C}_5^0(\theta, \phi) \right.$$
$$\left. - \ldots\right],$$

if $\rho > 1$, and is

$$V = M\left[\frac{3}{2}A_0 + \frac{3}{2}A_1\rho C^0_1(\theta, \phi) + A_2\rho^2 C^0_2(\theta, \phi) + \frac{3.1}{2.4}A_3\rho^3 C^0_3(\theta, \phi) \right.$$
$$\left. + \frac{3\,1.1}{2.4.6}\cdot A_5\rho^5 C^0_5(\theta, \phi) + \ldots\right],$$

if $\rho < 1$, $0 \leqq \theta \leqq \frac{\pi}{2}$, $0 \leqq \phi \leqq 2\pi$, where A_0, A_1, A_2, etc., are functions of a, b and c, the semi-axes of the ellipsoid $\rho = 1$, such that when $a = b = c = 1$, $A_0 = A_1 = A_2 = \ldots = 1$.

Let us next consider the potential of an ellipsoidal conductor at any outside point.

The potential at any outside point can obviously be written in the form

$$V = \sum_n A_n \frac{1}{\rho^{n+1}} \mathfrak{C}_n^o(\theta, \phi).$$

If the potential on the surface of the conductor be equal to unity, then

$$A_n = \int_0^\pi \int_0^{2\pi} \left[\mathfrak{C}_n^o (\theta, \phi) \right]^2 \frac{\sin \theta}{p_0^2} d\theta d\phi = \frac{1}{\mu_n},$$

that is to say, the potential at any outside point is given by

$$V = \sum_n \frac{1}{\mu_n \rho^{n+1}} \mathfrak{C}_n^o(\theta, \phi).$$

In a similar manner the potential of many other problems involving ellipsoidal boundaries can be investigated.

2. *Vibrations of a gas inside a rigid ellipsoidal envelope.*

One of the most interesting applications of the approximate solutions of the wave equation that we have obtained in Part I, is to the investigation of the motion of a gas within a rigid ellipsoidal envelope. To determine the free periods we have only to suppose that $\frac{\partial \Psi}{\partial \rho}$ vanishes when $\rho = 1$. The symmetrical vibrations in which the disturbance in each similar and similarly situated ellipsoidal surfaces is in the same phase will be determined by $\Psi_0 (k'\rho)$ which satisfies the equation.

$$\rho \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Psi_0}{\partial \rho} \right) + \frac{3}{\rho} \frac{\partial \Psi_0}{\partial \rho} + \frac{3k^2}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \Psi_0 = 0,$$

Therefore

$$\Psi_0 = \frac{\sin \left\{ \dfrac{\sqrt{3} k\rho}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)}} \right\}}{\rho},$$

The free periods are given by

$$\frac{2\pi\rho}{\lambda} = \frac{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)}}{\sqrt{3}} (1{\cdot}4303\pi,\ 2{\cdot}4590\pi,\ 3{\cdot}4709\pi,\ 4{\cdot}4774\pi,\ 5{\cdot}4818\pi,\ 6{\cdot}4844\pi, \text{ etc.}),$$

λ being the wave length.

The first finite root corresponds to the symmetrical vibration of the lowest pitch. In the case of a higher root the vibrations in question has ellipsoidal nodes defined by the values of ρ corresponding to the inferior roots. It will be noticed that the pitch would be lower for the ellipsoidal shell than for a corresponding spherical shell obtained by putting $a=b=c=1$. The amount by which the pitch is decreased for an ellipsoidal shell of given dimensions can be easily calculated from the above formula.

The case of $n=1$ is perhaps the most interesting. The differential equation satisfied by Ψ_1 is

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Psi_1}{\partial\rho}\right)+\frac{5}{\rho}\frac{\partial\Psi_1}{\partial\rho}+k'^2\Psi_1=0$$

where

$$k'^2=\frac{5k^2}{\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},\quad\frac{5k^2}{\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}}\text{ or }\frac{5k^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}},$$

Hence the vibration at any point is given by

$$-\frac{\partial}{\partial(k'\rho)}\frac{\sin k'\rho}{k'\rho}U_1.$$

where $U_1=a\sin\theta\cos\phi$, $b\sin\theta\sin\phi$, or $c\sin\theta$ The air therefore sways from side to side in the directions of the three principal axes. For vibrations in the direction of the a-axis, the periods are given by

$$\frac{2\pi}{\lambda}=\frac{\sqrt{\left(\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}{\sqrt{5}}\;(\cdot6625\pi,\ 1\cdot8908\pi,\text{ etc.})$$

for vibrations in the direction of the b-axis by

$$\frac{2\pi}{\lambda}=\frac{\sqrt{\left(\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}\right)}}{\sqrt{5}}\;(\cdot6625\pi,\ 1\cdot8908\pi\text{ etc.})$$

and for those in the direction of the c-axis by

$$\frac{2\pi}{\lambda}=\frac{\left(\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}}\right)}{\sqrt{5}}\;(\cdot6625\pi,\ 1\cdot8908\pi,\text{ etc.}),$$

λ as before denoting the wave length.

When $n=2$, the differential equation satisfied by Ψ_2 is

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Psi_2}{\partial\rho}\right)+\frac{7}{\rho}\frac{\partial\Psi_2}{\partial\rho}+k'^2\Psi_2=0,$$

where $k'^2 =$

$$\frac{7k^2\,[12a^4+3\,(b^4+c^4)-4a^2\,(b^2+c^2)+2b^2c^2]}{8(b^2+c^2+4a^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12a^4+3\,(b^4+c^4)-4a^2\,(b^2+c^2)+2b^2c^2\right]},$$

$$\frac{7k^2\,[12b^4+3\,(c^4+a^4)-4b^2\,(c^2+a^2)+2c^2a^2]}{8\,(c^2+a^2+4b^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12b^4+3\,(c^4+a^4)-4b^2\,(c^2+a^2)+2c^2a^2\right]},$$

or

$$\frac{7k^2\,[12c^4+3\,(a^4+b^4)-4c^2\,(a^2+b^2)+2a^2b^2]}{8\,(a^2+b^2+4c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12c^4+3\,(a^4+b^4)-4c^2\,(a^2+b^2)+2a^2b^2\right]^2}.$$

The spherical nodes are given by

$$\tan k'\rho = \frac{k'^3\rho^3-2k'\rho}{4k'^2\rho^2-9}$$

of which the first finite solution is $k'\rho=3{\cdot}3422$, giving a tone graver than any of the symmetrical group. The following will be seen to be nodal surfaces

$$2x^2-y^2-z^2=0,\quad 2y^2-z^2-x^2=0,\quad 2x^2-x^2-y^2=0.$$

It will appear from the above results that corresponding to a single mode of vibration of the gas inside a spherical shell we get three distinct mode of vibrations for the ellipsoidal shell. This result is also clear from the general expression. The periods of the nth mode are determined by k which are the roots of the equation

$$\frac{\partial}{\partial\rho}\left[\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\sin k'\rho}{\rho}\right]=0,$$

where

$$k'^2 = k^2\,.\,\frac{\int_0^\pi\int_0^{2\pi}[C_n^0(\theta,\,\phi)]^2\sin\theta\,d\theta d\phi}{\int_0^\pi\int_0^{2\pi}[C_n^0\,(\theta,\,\phi)]^2\,\frac{\sin\theta}{p_0^2}d\theta d\phi}$$

It is clear from the above expression that by an interchange of the letters a, b, c in the expression $C_n^0\,(\theta,\,\phi)$, we get three distinct types of vibration.

3. *Non-stationary state of heat in an ellipsoid.*

If the initial temperature of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

at any point x, y, z be $f(x, y, z)$ and its boundary is maintained at temperature zero, then the problem consists in the determination of the temperature $\Psi(x, y, z, t)$, such that

(1) $\frac{\partial\Psi}{\partial t}=\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}$ at every point inside the ellipsoid, the diffusivity being assumed to be unity,

(2) $\Psi=0$ on the boundary,

(3) $\Psi=f\ (x, y, z)$ when $t=0$.

A set of approximate solutions of the equation (1) in the co-ordinates (ρ, θ, ϕ) are

$$e^{-\lambda^2 t}\ \Psi_n\ ((\lambda\rho)\ C_n^m\ (\theta, \phi)$$

or

$$e^{-\lambda^2 t}\ \Psi_n\ (\lambda\rho)\ S_n^m\ (\theta, \phi).$$

Hence the solution of the problem is given by

$$\Psi(x, y, z, t)=\sum_{n,m} A_{n,m}\ e^{-\lambda^2 t}\ \Psi_n\ (\lambda\rho)\ C_n^m\ (\theta, \phi)$$

where the parameter λ is a root of the equation

$$\Psi_n\ (\lambda)=0$$

and if $f\ (x, y, z)\Rightarrow f\ (\rho, \theta, \phi)$, we have

$$A_{n,m}=\frac{\int_0^1\int_0^\pi\int_0^{2\pi} f\ (\rho, \theta, \phi)\ \Psi_n\ (\lambda\rho)\ C_n^m\ (\theta, \phi)\ \frac{\sin\theta^2}{p_0^{\,2}}\ d\rho d\theta d\phi}{\int_0^1 [\Psi_n\ (\lambda\rho)]^2\rho^2 d\rho\ \int_0^\pi\int_0^{2\pi}[C_n^m\ (\theta, \phi)]^2\ \frac{\sin\theta}{p_0^{\,2}}\ d\theta d\phi}$$

When $n=0$, the parameter λ is given by

$\lambda=S\pi.\ \frac{1}{\sqrt{3}}\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$, S denoting any positive integer.

If $b^2=c^2=a^2\ (1-e^2)$, e denoting the eccentricity of the ellipsoid of revolution, we have

$$\gamma a=S\pi\ \frac{1}{\sqrt{3}}\left(1+\frac{2}{1-e^2}\right)^{\frac{1}{2}}=S\pi\ \frac{1}{\sqrt{3}}\left[1+2\ (1+e^2)\right]^{\frac{1}{2}}$$

$=S\pi\,(1+\frac{1}{3}e^2)$, neglecting the fourth and the higher powers of e^2. This agrees with the result obtained by Niven.*

If $b^2=a^2\,(1-e_1^{\,2})$ and $c^2=a^2/(1-e_2^{\,2})$, so that e_1 and e_2 are the eccentricities of the principal diametral planes, we have

$$\lambda a=S\pi\,\frac{1}{\sqrt{3}}\,\sqrt{}\left(1+\frac{1}{1-e_1^{\,2}}+\frac{1}{1-e_2^{\,2}}\right)^{\frac{1}{2}}$$

$$=S\pi+\tfrac{1}{6}\,S\pi\,(e_1^{\,2}+e_2^{\,2}),$$

neglecting the fourth and the higher powers of e_1 and e_2.

4. *Scattering of a set of plane waves by an ellipsoidal obstacle.*

Let us next consider the problem of the scattering of a set of plane sound waves by an ellipsoidal obstacle.

The velocity potential of a set of plane waves proceeding in the direction ($\sin u\cos v$, $\sin u\sin v$, $\cos u$) can be expressed in the form

$$\phi_0=e^{ik(x\sin u\cos v+y\sin u\sin v+z\cos u)}$$

$$=e^{ik\rho(a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u)}+$$

$$\sum_{n=0}^{\infty}\frac{(ik\rho)^n}{2\lambda_n}\Psi_n\,(k_n\rho)\,C_n(a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u)$$

approximately, where

$$k_n^{\,2}=k^2\,\frac{\displaystyle\int_0^{\pi}\!\!\int_0^{2\pi}[C_n^{\,o}\,(\theta,\phi)]^2\sin\theta\,d\theta d\phi}{\displaystyle\int_0^{\pi}\!\!\int_0^{2\pi}C_n^{\,o}\,[(\phi,\theta)]^2\,\frac{\sin\theta}{p_0^{\,2}}\,d\theta d\phi}$$

The expression for ϕ_0 can also be written in the form

$$\phi_0=\sum_{n=0}^{\infty}\frac{(ik\rho)^n}{2\mu_n}\Psi_n\,(k_n\rho)$$

$$\left[C_n^{\,o}\,(\theta,\phi)\;P_n\,(\cos u)+2\sum_{m=1}^{n}\frac{(n-m)\,!}{(n-m)\,!}\left\{C_n^{\,m}\,(\theta,\phi)\;P_n^{\,m}\,(\cos u)\cos mv\right.\right.$$

$$\left.\left.+\,S_n^{\,m}\,(\theta,\phi)P_n^{\,m}\,(\cos v)\sin mv\right\}\right].$$

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* *Phil. Trans.*, Vol. 171 (1880), p. 145.

The proof of this expression depends on the fact that if we make the wave proceed in the direction of the z-axis, that is we make $u=0$, we have

$$\phi_0 = e^{ik\rho\, c \cos\theta} = \sum_{n=0}^{\infty} \frac{(ik\rho)^n}{2\lambda_n} \Psi_n (k_n\rho)\, C_n^o \;(\theta.\ \phi)$$

and this is approximately true for

$$\int_0^{\pi}\int_0^{2\pi} e^{ik\rho\, c\cos\theta}\, C_n^o\,(\theta,\ \phi)\, \frac{\sin\theta}{p_0^2}\, d\theta d\phi = 2\,(ik\rho)^n\, \Psi_n'\,(k_n\rho)$$

approximately.

Consider now a constituent

$$\phi_0 = B_n^m\, \rho^n\, \Psi_n\,(k_n\rho)\;\; C_n^m\,(\theta,\ \phi)$$

of the incident wave system, and let the corresponding constituent of the scattered waves be

$$\phi_1 = B_{1n}^m\, f_n\,(k_n\rho)\, C_n^m\,(\theta,\ \phi),$$

where,

$$f_n\,(\rho) = \left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \frac{e^{-i\rho}}{\rho},$$

and as before

$$k_n^2 = k\; \frac{\int_0^{\pi}\int_0^{2\pi} \left[\, C_n^o\,(\theta,\ \phi)\right]^2 \sin\theta\, d\theta d\phi}{\int_0^{\pi}\int_0^{2\pi} \left[\, C_n^o\,(\theta,\ \phi)\right]^2 \frac{\sin\theta}{p_0^2}\, d\theta d\phi}.$$

If the ellipsoidal obstacle be fixed, the condition

$$\frac{\partial}{\partial\rho}\,(\phi_0 + \phi) = 0.$$

to be satisfied for $\rho = 1$, gives

$$\frac{B'^m_n}{B_n^m} = -\,\frac{k_n\Psi'_n\,(k_n) + n\Psi_n\,(k_n)}{k_n\, f'_n\,(k_n) + n\, f_n\,(k_n)}$$

When the wave length is large compared with the dimensions of the ellipsoid, we have

$$\frac{B'^{m}_{n}}{B^{m}_{n}} = \frac{n}{n+1} \cdot \frac{k_n 2n+1}{\{1.3\ldots(2n-1)\}^2 . (2n+1)}$$

approximately, and

$$\frac{B'_0}{B_0} = -\tfrac{1}{3} k_0{}^3 = -\frac{k^3}{3}\left(\frac{3}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\right)^{\frac{3}{2}}$$

approximately.

We have $$B^{m}_{n} = \frac{(n-m)!}{(n+m)!} i^n k^n \frac{1}{\lambda_n} P^{m}_{n}(\cos u) \cos mv.$$

The rate at which energy is scattered can be easily shown to be

$$\frac{2\pi\mu_n}{k_n 2n+2} \left| B'^{m}_{n} \right|^2$$

and by considering the terms of the lowest order it can be shown that it varies as the fourth power of the wave length. This is true whatever be the shape of the body so long as the dimensions of the body are small compared to the wave length.*

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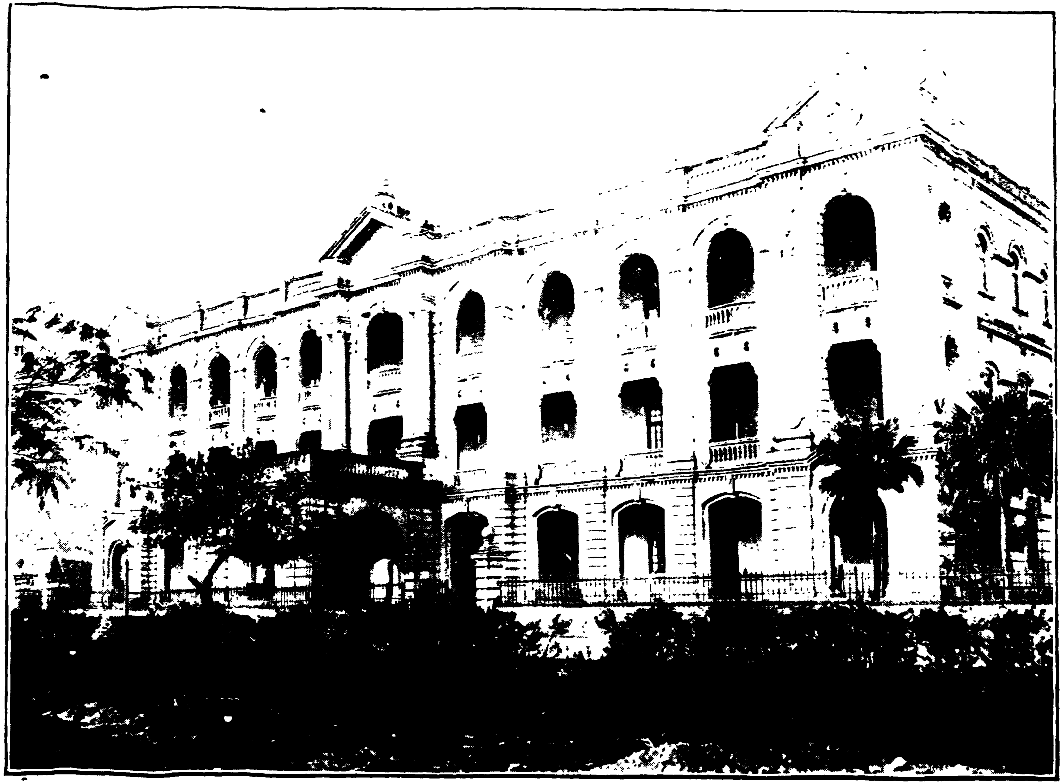
* Rayleigh, *Scientific Papers*, Vol. I, pp. 91–92.



The University College of Science, Calcutta.

THE UNIVERSITY COLLEGE OF SCIENCE AND ITS ACTIVITIES*.

Introductory.

The laying of the foundation-stone of the Palit Laboratories of the University College of Science at Calcutta by Sir Asutosh Mookerjee on the 27th of March, 1914, may be justly said to mark a new epoch in the history of scientific education in India. That event was made possible by the generosity of two men whose names will pass down to history, Sir Taraknath Palit and Sir Rashbehari Ghosh, and no less so by the energy and enthusiasm of Sir Asutosh Mookerjee himself, manifested in a quarter of a century's devoted efforts in advancing the cause of education and preparing the ground for the establishment of an institution where his aims would find their fullest achievement. Indeed it would be strictly accurate to say that it was the character and abilities of Sir Asutosh himself, the belief in the sanity and soundness of his ideals, in his untiring energy and administrative capacity and in his extensive academic experience and knowledge, quite as much as the great object of the proposed institution, that evoked the generosity of Sir Taraknath Palit and Sir Rashbehari Ghosh towards the Calcutta University. A perusal of the two trust-deeds in which Sir Taraknath Palit made over the whole of his property, and of the letters of Sir Rashbehari Ghosh offering his gifts for the foundation of the chairs in Pure and Applied Science and for the maintenance of a department of technology will make this entirely clear. Our profound thankfulness to the two great donors whose statues grace the portico of the college building is enhanced by our appreciation of their discernment. It has been truly said that the memory of one who gives to learning lives on when kings and emperors have been forgotten. When future generations recall the gifts of Sir Taraknath Palit and Sir Rashbehari Ghosh, they will also recall the name of Sir Asutosh Mookerjee as the founder of the institution associated with their benefactions and as one who devoted his life to the advancement of learning in his country.

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* This article has been compiled by the various heads of each department. The *Introductory note is by Dr. C. V. Raman, M.A., D.Sc., Palit Professor of Physics, in the* College Science, Calcutta.

The significance of the foundation of the University College of Science at Calcutta lay in the fact that it was here for the first time since the establishment of the Indian Universities in the fifties of the last century that higher scientific teaching in all its branches was taken up by one of them as part of its regular functions, and that the advancement of knowledge by scientific research was recognised as one of the chief duties of the teacher. The Sir Taraknath Palit Professorship of Chemistry and Sir Taraknath Palit Professorship of Physics were the two first University Chairs in Science to be founded in India. One of the conditions imposed by the donors was that all the chairs founded from their benefactions should be exclusively held by Indians. The significance of this will be appreciated from the fact that at the time the benefactions were made, Indians were practically barred from entry into the higher educational services under Government. The University College of Science was in fact founded to offer Indians opportunities for exercise of initiative and achievement which were denied to them elsewhere. Lecturerships were established for many different branches and sub-departments of science, especially those of recent development, no attempts for teaching which had been previously made in the colleges affiliated to the University, or for the matter of that anywhere else in India. It is very regrettable that these magnificent efforts at self-help and progress made by the Calcutta University in the matter of scientific teaching and research have hitherto evoked absolutely no recognition or assistance from the educational administrators at Simla and Darjeeling.

The imposing building at 92, Upper Circular Road, Calcutta, of which a photograph is here reproduced, houses the departments of Physics, Chemistry and Applied Mathematics and the attached workshops. Some rooms have also been allotted for Experimental Psychology and Bio-chemistry. The private residence of Sir Taraknath Palit at Ballyganj, a few miles off, with the spacious grounds in which it is situated, have been converted into botanical and zoological laboratories with quarters for the Professors and students. Separate provision has also been made for teaching in Geology, Anthropology and Physiology. The writer does not feel competent to give a detailed account of all the activities of the University College of Science since its foundation, and even if he could, would not perhaps be doing full justice to all, especially in view of the fact that some of the departments were only started recently under considerable difficulties owing to the conditions created by the war. As is evident from a perusal of the reports of the Council of Post-Graduate Teaching in Science for the past three years, the organization of higher scientific teaching has now been developed in the Calcutta University to an extent not so far attempted anywhere else in India, and the beneficial results are already becoming evident in the numerous research papers being contributed by the various members of the University staff and their scholars.

The Department of Physics.

Though the Sir Taraknath Palit Chair of Physics was offerred to the writer in 1914, owing to the law-suits on the endowment and other difficulties, it was possible for him to join it only in July 1917. Partly owing to war-conditions and partly owing to the financial difficulties of the University, serious hindrances were felt in organizing the work of the department, especially in obtaining apparatus for teaching and research in the newer branches of the subject. Nevertheless, some progress has been made, and writing now after three years and a half of work, it is permissible to look back and recount the broad aspects of the results.

Perhaps the most encouraging result of all is that the department has produced a number of men who have succeeded in proving their capacity for successful research in Physics with only such training and facilities as are available in India. Four members of the staff of the Physics department have succeeded in obtaining their Doctorate during this period. Another obtained the Curzon Research Prize of the Madras University and was selected for an important and responsible charge in the Indian Meteorological Department. At least a dozen others connected in one way or another with the department have published research work in various journals during the period under review, and some of them are men of high promise of whom more will be seen and heard in the years to come. All this is legitimately a matter for satisfaction.

The activities of the department have been many-sided in their nature. During the period under review, research work was being carried on practically in all the branches of Physics, and papers dealing with a very wide range of subjects have been published. In the department of Optics and Spectroscopy no fewer than fifty papers were contributed, as also ten papers in General Physics, three in Heat, one in Magnetism, and some twenty papers in Acoustics, by the writer and those associated with him during the period under review. As time goes on and experience is gained, an increasing measure of success may be expected, and there is room for hope that at least in some branches of Physics, the University College of Science at Calcutta may obtain a leading position in the scientific world as a centre of research.

The Department of Applied Mathematics.

The principal features of the activities that have marked the few years the department has been in existence as an integral part of the University College of Science may here be set forth in brief. The department is composed of the Sir Rashbehari Ghose Professor of Applied Mathematics, three University Lecturers, two Sir Rashbehari Ghose Research Scholars and several students who are either preparing for the Master of Arts or Science degree of the University or after having taken those degrees are engaged in post-graduate research. Contributions have been made by the members of the staff and the research students on various branches of applied mathematics, namely, theory of attraction and potential, theory of vibrations, harmonics (including Bessel and Mathieu functions), motion of solids through liquids, electrical, optical, acoustical, tidal and earthquake wave motion and astrophysics. Some of the researches were undertaken by their authors with a view either to explain or elucidate some new physical phenomena actually observed in the course of experimental work in the Physical Laboratory. In this way the Departments of Physics and Applied Mathematics have been closely co-operating with each other. Some of the researches in applied mathematics have been published in foreign journals but they have been mainly published in the *Bulletin of the Calcutta Mathematical Society* which is the quarterly organ of the Society, at present running through the twelfth volume: The Sir Rashbehari Ghose Professor of Applied Mathematics who is the Secretary of the Society and the Editor of the journal is at present aiming at its further development and improvement and it is hoped that ere long it will be recognised as one of the leading mathematical journals of the world and will do away with the reproach that has been rightly or wrongly levelled by some of its critics against the scientific workers of the University that their conrtibutions published in foreign journals benefit the foreigners more than their own countrymen.

The Department of Chemistry.

The Department of Chemistry was the first to be inaugurated in 1915, with Sir P. C. Ray and Dr. P. C. Mitter as the first professors. A number of junior workers joined, who have since distinguished themselves by their discoveries in the domain of pure chemistry. The articles in this volume, by the various workers of this department, are ample proof of the excellent work being carried on here. The facilities afforded to the students are of the highest order. Post-graduate teaching is the only form of teaching which the Institution undertakes. The Professors and University Lecturers take direct and personal interest in the work of the students.

Every professor is provided with two research scholars, their stipends being met from the endowments of the late Sir Tarak Nath Palit and the late Sir Rashbehari Ghose.

In 1920, the Department of Applied Chemistry was added through a second munificence of the late Sir Rashbehari Ghose. Dr. H. K. Sen, after spending six years in industrial work outside, has returned to Calcutta to fill the first chair of Applied Chemistry. Twenty-six students have been admitted to this department. Students are taken in batches to visit factories, where teachers are present to explain the details of working. The instruction in this department will be substantially improved when the workshop, the construction of which was undertaken a few months ago, is completed. This will include machines and other accessories that are in use in factories. There is a scheme for installing model working units of the more important industries that are vital to the country. This department has in addition a lecturer on mechanical drawing.

The Department of Experimental Psychology.

The term 'science' is associated in the popular mind, with matter It is concerned with changes in the physical world and it puts man in a position to shape and control the realm of matter in the fashion that his interests dictate. It is but to be expected then that for most people it would seem to be an anomaly to see the Department of Experimental Psychology located in the College of Science.

But the realm of mind too, is amenable to laws, the psychic events also are capable of prediction and control, as are the events of realm of matter and of the physical world. Psychology has thus a very good claim to a place in the congress of sciences. And the authorities of the Calcutta University are to be congratulated upon their taking this broad view of science.

Many, however, will hesitate to assent. Is not the spirit free from all determinations and is not its law the law of spontaneity? Thus will the eternal Metaphysic enter its challenge and protest. But Psychology is not concerned with the ultimate nature of the spiritual reality. We know that we frequently 'explain' our fellow-men's behaviour; and no explanation is possible unless there are generally valid laws. We are able to predict our neighbour's conduct; or else, all social intercourse would be at an end. We believe that we can control human mind. What else are the processes of education and training but so many ways of 'controlling' the mind? Why else does the business man advertise his wares if not for 'controlling' the public mind in his own favour? Mental events then can be 'explained', 'predicted' and 'controlled' as much as physical events. And the 'Science of Mind' has its own place in the scheme of natural sciences. We might even say that Psychology is the latest step forward by the Spirit of Science. It views mind in the same manner as the physical sciences view the physical world. The same attitude of determinism is the ruling attitude everywhere.

As a science Psychology is but in its youth, if not in its infancy. Yet it promises a precocious future. Already there are signs of its far-reaching influence, potential if not actual, in many domains of human enterprise. We should be failing in our cultural zeal if we do not welcome this new agency of progress.

What has this science acheived in the arena of practical life? Its achievement, it is true, cannot be pointed out to be stared at by the bystander. But to the thinking mind it is not obscure. Its influence upon education is not far to seek. The great educationists from Rousseau onwards have sought a solid psychological foundation for this greatest of human problems, that of Education. Their attempts have met with varied success, because the science of mind in those days was yet to be born. All educational movements of to-day feel their strength to lie exactly upon this basis—an accurate knowledge of the growing mind.

But recent years have shown many other lines of operation. The upheaval of the last war forced all the progressive nations to marshall their best human material for the multiform national needs and activities. It involved choice and rejection, allocation of each unit to its proper place. And psychology came forward to help; for selection of man is nothing but selection of his mind,—testing minds to discover what they are best fitted for. But nowhere has the harvest been so rich as in the studies of mental diseases. Methods of diagnosis and treatment have reached a degree of elaboration hitherto unknown. The numerous studies that are coming to light unerringly point to a great future. This is as was to be expected; for, the practical application does not far linger behind the discovery of the law.

The Laboratory at the College of Science is devoted to this new science in its many-sided development. It is as yet mainly a teaching laboratory; for a science in order to flourish and progress must have its own human atmosphere. And this atmosphere has to be created. We are making a persistent effort to foster interest in almost all the special lines into which Psychology has branched off. We are trying to help in the forward march of the science so far as circumstances permit.

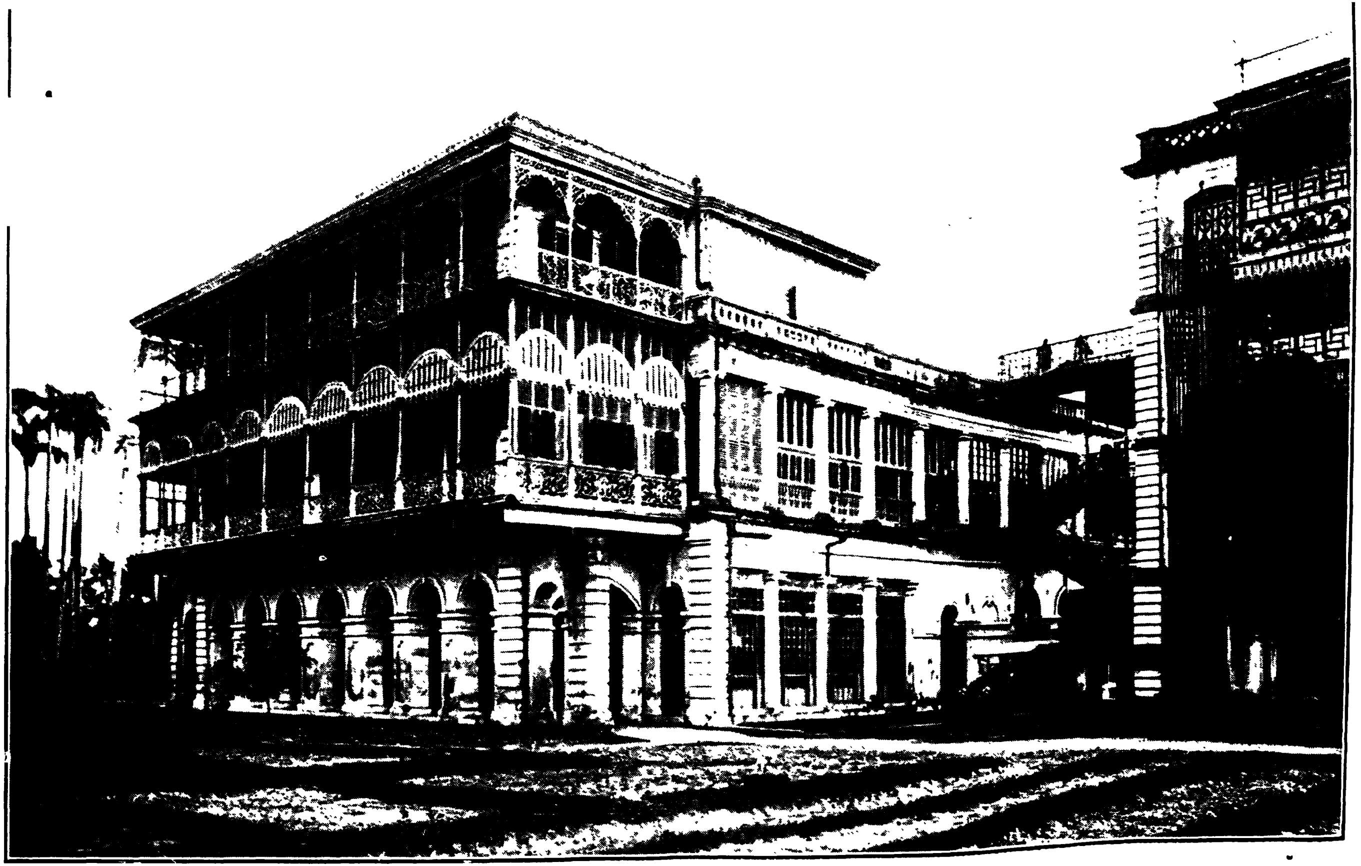
The Laboratory is about four years old; it has yet to take to soil. But we have already a band of workers who, though they cannot claim remarkable achievements, can claim earnestness of purpose and determination to work. Studies are in progress in several directions. One of the teachers has been working with success in the field of Abnormal Psychology and has reported some of his results which have been adjudged to be of value. Two others are engaged in the interesting problem of studying certain psycho-physical capacities of our college students; and a report of a portion of this appears in this volume. Another member of the staff has interested himself in certain problems of General Psychology and has given out a portion of his work. Besides these, researches are in progress in several important subjects and will see light before long.

There are many fields of work open to us. Education, as the problem of abiding interest, will largely claim our attention. Every race must discover and understand its own mind and must shape its own plan of training. It is through an elaborate survey of the different mental capacities—of fatigue, of reaction, of attention, of memory, etc.—of the adult and of the growing mind that we can make our educational system fruitful and sound. The problems of backwardness and of 'mental deficiency' in their various orders will have to be worked out and will lead us to the study of individual and class differences. Much can be expected from a psychological survey, at least of the school-going population. There are other topics equally promising. A study of the criminal types, more particularly, a psychological examination of juvenile offenders, would yield results valuable alike for sociological and for educational purposes. Abnormal Psychology is a subject by itself and offers a rich field for those who propose to study the many forms of aberrations that are to be found in every social order. We have, then, before us the task of formulating psychological tests for different purposes, vocational and educational. Besides these, the problems of General Psychology that can be fruitfully investigated are too various to enumerate. We have, thus, a hopeful programme before us. We have not done much; we look forward to the future,—not a very distant one.

The progress of a science often depends upon the public interest that it can command, which again largely follows the utilitarian possibilities. We are convinced that we can make our science useful to the community; and we want the community to take interest in it.

The Departments of Biology.

The residence of the late Sir Taraknath Palit has been fitted up to house the Botanical and Zoological Departments. The work of reconstruction was commenced in 1916 and the Botanical Department was opened in October 1918. Mr. S. Bal, M.Sc., of the Michigan University, has been working up the mycological branch and has published eleven papers in the Science Journal of the University; a paper on a species of parasitic algae appears in this volume. Several new species of fungi are about to be published under the joint authorship of Dr. H. Sydow of the Berlin University and Professor Bal. A plot of land is being devoted to the cultivation of Indian medicinal



Biological Laboratory, Palit House, Ballygunj.

plants under Prof. Bal's direction. An immense amount of work has been done on medicinal plants in recent years both in Europe and America, and considering that about seventy per cent of the medicinal plants recognised by the British Pharmacopœa are found in India, it is high time that work of that kind be seriously taken in hand in this country, and it is to be hoped that funds for this class of research will be forthcoming.

Dr. S. P. Agharkar, the Rashbehari Ghosh Professor of Botany, returned from Europe in July 1920, after completing his studies at the University of Berlin and a botanical tour through Southern France and Spain. He had the good fortune to be presented with five hundred named specimens by the authorities of the Berlin Botanical Museum, and further with a valuable set of *Characeae*, a gift by Mr. Groves, and has been promised a collection of Norwegian plants by Prof. Wille, of Christiania. The most important addition to the herbarium of the University College of Science is a collection of about five hundred specimens of plants from the interior of Nepal made during a journey undertaken by Dr. Agharkar during the months of June and July of last year. This collection is of great interest as it was made within a zone of from 4,000 to 14,500 feet visited by Dr. Wallich's collectors many years ago. Although the department presided over by Dr. Agharkar is principally intended to serve the purposes of Applied Botany, the absence of a farm and the want of funds are proving great obstacles, the aims of Dr. Agharkar being to study the relationship of plants of importance in agriculture, to investigate the diseases of economically important plants, their causes and prevention, to discover by experiment and selection new races and to carry out experiments, in co-operation with the Bengal Agricultural Department, experiments on plant-breeding. Dr. Brühl is working out the materials for local floras of Bengal and Sikkim, is conducting, in colaboration with one of the Post-graduate students and with the assistance of members of the Sanitary and the Fisheries Departments, a detailed examination of the Algae Flora of Bengal with special reference to the algae found in filter beds and those of great importance in pisciculture, and, in colaboration with a member of the staff of the department of zoology, is engaged in researches on plant galls. The instrumental equipment for botanical research of the highest order is available, the scheme for a systematic botanic garden has been worked out and its speedy realisation is only a matter of the necessary funds being forthcoming.

The Zoological Department was opened in 1919, with a library of about 850 volumes and a laboratory which has gradually developed. The museum is still in its infancy; Mr. Srinivasa Rao, now lecturer on zoology, has made a collection of local insects, whilst other material has been kindly lent by the authorities of the zoological department of the Indian Museum. Professor Mitter has published a treatise in Bengali, মনের বিবর্তন (*Moner Bibartan*), and is preparing another Bengali treatise on the theory of evolution and evidences in its support; he is also engaged in a research into the part played by olfactory sensation of ants aiding in the selection of food. Professor Maulik, whilst in London and before joining his appointment as University Professor of Zoology, worked out the subfamilies Hispinae and Cassidinae, members of the Order of Coleoptera, for the Fauna of British India. He has left again for England.

Original research in the zoology department is greatly hindered by the necessity of opening classes for B.Sc. students who wish to specialize later on in zoology.

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